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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

(22 ALBEMARLE STREET, W.)

VOL. XXXIV.—NOS. 764—766.

Communications for the Secretaries may be forwarded to them at the following addresses:—

London Mathematical Society, 22 Albemarle Street, W. } R. TUCKER.
24 Hillmarton Road, West Holloway, N. }
34 St. Margaret's Road, Oxford.—A. E. H. LOVE.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contributions which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of ten guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting held on the second Thursday in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the year 1901 are the *second* Thursdays in January, February, March, April, May, June, November, and December.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

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OF

THE LONDON MATHEMATICAL SOCIETY.

VOL. XXXIV.

THIRTY-SEVENTH SESSION, 1900-1901

(since the Formation of the Society, January 16th, 1865).

Thursday, March 14th, 1901.

Dr. HOBSON, F.R.S., President, in the Chair.

Twelve members present.

Messrs. Harris Hancock, Professor of Mathematics in the University of Cincinnati, U.S.A., and Alfred William Porter, B.Sc., Assistant Professor of Physics in University College, London, were elected members.

Sir Robert Ball was admitted into the Society.

Prof. Elliott gave an account of "Some Algebraical Identities of Simple Arithmetical Application."

Prof. Love gave a "Preliminary Notice concerning the Theory of Stability of Motion." The Chairman, Mr. Macdonald, and Lt.-Col. Cunningham joined in a discussion of this last contribution.

Papers by Prof. Burnside "On the Composition of Group-Characteristics," and by Mr. G. H. Hardy "On the Elementary Theory of Cauchy's Principal Values," were communicated by reading their titles.

The following presents were made to the Library :—

"Educational Times," March, 1901.

"Indian Engineering," Vol. xxix., Nos. 4-7, Jan. 26-Feb. 16, 1901.

The following exchanges were received :—

"Transactions of the Royal Society," Vols. cxcii.-cxcv., 1901.

"Proceedings of the Royal Society," Vol. lxviii., No. 442 ; 1901.

"Royal Society : Applications, 1901, for Government Grant for Scientific Investigation" ; 1901.

"Bulletin of the American Mathematical Society," Series 2, Vol. vii., No. 5, February ; New York, 1900.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. xiv., No. 3 ; Coimbra, 1900.

"Bulletin des Sciences Mathématiques," Tome xxiv., Nov., 1900 ; Paris.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. vii., Fasc. 1, Jan., 1901 ; Napoli.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. x., Fasc. 3, 4 ; Roma, 1901.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," Bd. lii., No. 7 ; 1900.

"Supplemento al Periodico di Matematica," Anno iv., Fasc. 3, 4 ; Livorno, 1901.

The following off-prints have been received from the Meteorological Office :—

"Sur quelques généralisations d'une relation appliquée par Hamilton et Mannheim," par M. Pelišek (Tr. "Mathematicko-přírodovědecka," 1900, ii.).

"Uber ein Analogon der Euler'schen Zahlen," von F. J. Studnička, ix., March, 1900.

"Entwickelungen einiger zahlentheoretischer Functionen in unendliche Reihen," von F. Rogel, xxx., Juli 1900.

"Zur rechnerischen Behandlung der Axonometrie," von J. Sobotka, xxxiii., October, 1900.

(The last three from the "Mathematisch-naturwissenschaftliche Classe" der Kön. Böhm. Gesellschaft der Wissenschaften in Prag.)

Presented by R. Tucker, Hon. Sec. :—

"Handbuch der Mathematik," herausg. von Dr. Schlömilch, unter mitwirkung von Prof. Dr. Reidt und Prof. Dr. Heger, Band ii., mit 235 holzschnitten.

"Analytische Geometrie," "Differential- und Integralrechnung, und Ausgleichungsrechnung," "Reuter-, Lebens- und Aussteuer-Versicherung" (all by Dr. Heger), Breslau, 1881, half calf binding.

A Class of Algebraical Identities and Arithmetical Equalities.

By E. B. Elliott. Read and received March 14th, 1901.

1. An arithmetical equality of universal application may or may not have as its basis a fact of algebraical identity. For instance, one which has is Gauss's

$$n = \Sigma\phi(d),$$

where the numbers d are the divisors of n, and, for an n whose distinct prime factors are $p, q, \ldots, t$, $\phi(n)$ denotes Euler's indicator $n\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\ldots\left(1-\frac{1}{t}\right)$, which is the expression for the number of numbers prime to and not exceeding n. This is an identity in $p, q, \ldots, t$ when n and the divisors d are given their expressions as products of primes—a different identity of course for one n from what it is for another of different algebraic form as a product of primes. It is just possible that the class of identities which follows, productive of a class of arithmetical equalities including the one above mentioned, may have hitherto escaped notice.

Let $p_1, p_2, p_3, \ldots$ be a set—either finite or infinite—of distinct symbols which obey the ordinary laws of algebraic combination, and let $p_r, p_s, p_t, \ldots$ be any chosen finite selection from them; and in a product such as

$$\Pi\left(1-\frac{1}{p}\right)$$

suppose that there is a factor corresponding to each of the whole set $p_1, p_2, p_3, \ldots$. Denote by

$$F_m(p_r^\rho p_s^\sigma p_t^\tau \ldots),$$

where $\rho, \sigma, \tau, \ldots$ are positive integers, the part of the direct expansion of

$$p_r^\rho p_s^\sigma p_t^\tau \ldots \Pi\left(1-\frac{1}{p}\right)^m$$

in descending positive zero and negative powers of the p's which involves no negative power of any p—the integral part, let us say. It is also of course the integral part of

$$p_r^\rho p_s^\sigma p_t^\tau \ldots \left(1-\frac{1}{p_r}\right)^m \left(1-\frac{1}{p_s}\right)^m \left(1-\frac{1}{p_t}\right)^m \ldots;$$

and for the values 0, 1, at any rate, of m is the whole of this product. The parameter m of an F_m is for the present unrestricted.

The following is an identity for any m:—

$$F_{m-1}(p_r^\rho p_s^\sigma p_t^\tau \ldots) \equiv \Sigma F_m(p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'}), \tag{A}$$

where the summation covers all combinations of integral and zero values of ρ', σ', τ', ... such that

$$0 \leqq \rho' \leqq \rho, \quad 0 \leqq \sigma' \leqq \sigma, \quad 0 \leqq \tau' \leqq \tau, \quad \ldots .$$

We have in fact

$$p_r^\rho p_s^\sigma p_t^\tau \ldots \Pi \left(1-\frac{1}{p}\right)^{-1} \equiv \Sigma\,(p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \ldots) + \text{fractional terms}; \tag{1}$$

and in this identity the integral parts of the two sides must be identical. This gives at once the case of (A)

$$F_{-1}(p_r^\rho p_s^\sigma p_t^\tau \ldots) \equiv \Sigma F_0(p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \ldots).$$

Now multiply (1) by $\Pi \left(1-\dfrac{1}{p}\right)^m$, thus obviously getting

$$p_r^\rho p_s^\sigma p_t^\tau \ldots \Pi \left(1-\frac{1}{p}\right)^{m-1} \equiv \Sigma \left\{ p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \ldots \Pi \left(1-\frac{1}{p}\right)^m \right\}$$
$$+ \text{fractional terms}.$$

The identification of the integral part of the left in this with that of the right gives (A) in its generality.

2. It is also possible, for any m, to express F_{m+1} linearly in terms of F_m's. From

$$p_r^\rho p_s^\sigma p_t^\tau \ldots \Pi \left(1-\frac{1}{p}\right) \equiv p_r^\rho p_s^\sigma p_t^\tau \ldots \left\{ 1 - \left(\frac{1}{p_r} + \frac{1}{p_s} + \ldots\right) + \left(\frac{1}{p_r p_s} + \ldots\right) - \left(\frac{1}{p_r p_s p_t} + \ldots\right) + \ldots \right\}$$
$$+ \text{fractional terms}$$
$$\equiv \Sigma \mu\,(p_r^{\rho-\rho'} p_s^{\sigma-\sigma'} p_t^{\tau-\tau'} \ldots)\, p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \ldots$$
$$+ \text{fractional terms}, \tag{2}$$

where $\mu\,(p_r^{\rho''} p_s^{\sigma''} p_t^{\tau''} \ldots)$ stands for 0 if any one of ρ'', σ'', τ'', ... exceeds 1, and for 1 or -1 if none of them exceeds 1, according as the number of them equal to 1 is even (zero reckoned even) or odd,

we obtain at once by identification of integral parts

$$F_1(p_r^\rho p_s^\sigma p_t^\tau \ldots) \equiv \Sigma\mu\,(p_r^{\rho-\rho'} p_s^{\sigma-\sigma'} p_t^{\tau-\tau'} \ldots)\, F_0(p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \ldots).$$

Now multiply (2) by $\Pi\left(1-\dfrac{1}{p}\right)^m$, and then use the fact that the integral parts of the identical expansions of the two sides must be themselves identical. The result is that, for any m,

$$F_{m+1}(p_r^\rho p_s^\sigma p_t^\tau \ldots) \equiv \Sigma\mu\,(p_r^{\rho-\rho'} p_s^{\sigma-\sigma'} p_t^{\tau-\tau'} \ldots)\, F_m(p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \ldots). \quad \text{(B)}$$

3. We can now apply arithmetically the identities (A) and (B). Let $p_1, p_2, p_3, \ldots$ be all the prime numbers 2, 3, 5, ..., and let $p_r, p_s, p_t, \ldots$ be those of them which are factors of a number

$$n = p_r^\rho p_s^\sigma p_t^\tau \ldots .$$

The numbers $p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \ldots$ are the divisors d, each once, of n. The central F, $F_0(n)$, of n is n itself. $F_1(n)$ is the indicator

$$\phi(n) = n\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{p_s}\right)\left(1-\frac{1}{p_t}\right)\ldots$$

of n. $F_{-1}(n)$ is the ordinary expression

$$\psi_1(n) = n\,\frac{1-\dfrac{1}{p_r^{\rho+1}}}{1-\dfrac{1}{p_r}}\;\frac{1-\dfrac{1}{p_s^{\sigma+1}}}{1-\dfrac{1}{p_s}}\ldots$$

for the sum of the divisors of n. The above has proved, by consideration of algebraical identities, a class of arithmetical equalities, of which one is the well known

$$n = \Sigma\phi(d),$$

and of which the general expression is

$$F_{m-1}(n) = \Sigma F_m(d), \quad \text{(C)}$$

and also the reversed class

$$F_{m+1}(n) = \Sigma\mu\left(\frac{n}{d}\right) F_m(d), \quad \text{(D)}$$

where $\mu(s)$ has its ordinary arithmetical definition as 0, or 1, or -1 according as s has a square factor, or an even (including zero), or an odd, number of unrepeated prime factors.

The law of formation of $F_m(n)$ needs no restatement. As examples of it we may write down, for instance,

$$F_3(p^3q^2r) = p^3q^2r\left(1-\frac{1}{p}\right)^3\left(1-\frac{1}{q}\right)^2\left(1-\frac{2}{r}\right),$$

$$F_{-2}(p^3q^2r) = p^3q^2r\left(1+\frac{2}{p}+\frac{3}{p^2}+\frac{4}{p^3}\right)\left(1+\frac{2}{q}+\frac{3}{q^2}\right)\left(1+\frac{2}{r}\right),$$

$$F_{-\frac{1}{2}}(p^3q^2r) = p^3q^2r\left(1+\frac{1}{2p}+\frac{3}{8p^2}+\frac{5}{16p^3}\right)\left(1+\frac{1}{2q}+\frac{3}{8q^2}\right)\left(1+\frac{1}{2r}\right).$$

4. We can readily write down sums for which the expressions are $F_m(n)$ for negative integral values of m. Whatever m be in (C), or say in

$$F_m(n) = \Sigma F_{m+1}(d),$$

we may replace, by the same law, each $F_{m+1}(d)$ by $\Sigma F_{m+2}(\delta)$ for all divisors δ of d. In the double sum obtained δ is in turn every divisor of n, and $F_{m+2}(\delta)$ occurs as many times as there are divisors d of n which δ divides, *i.e.*, $\psi_0\left(\frac{n}{\delta}\right)$ times, where ψ_0 means the number of divisors of its argument. Thus, changing notation,

$$F_m(n) = \Sigma F_{m+2}(d)\,\psi_0\left(\frac{n}{d}\right)$$

for all divisors d of n.

Let us now replace each $F_{m+2}(d)$ by the corresponding $\Sigma F_{m+3}(\delta)$. We deduce, writing d in place of δ, that

$$F_m(n) = \Sigma F_{m+3}(d)\left\{\psi_0\left(\frac{n}{d_1}\right)+\psi_0\left(\frac{n}{d_2}\right)+\dots\right\},$$

where d is in turn each divisor of n, and, for each d, the numbers $d_1, d_2, \dots$ are the $\psi_0\left(\frac{n}{d}\right)$ divisors of n which have d for a divisor, *i.e.*, $\frac{n}{d_1}, \frac{n}{d_2}, \dots$ are the $\psi_0\left(\frac{n}{d}\right)$ divisors of n whose conjugates have d for a divisor, *i.e.*, the $\psi_0\left(\frac{n}{d}\right)$ divisors of $\frac{n}{d}$. Write this

$$F_m(n) = \Sigma F_{m+3}(d)\,\psi_0^{(2)}\left(\frac{n}{d}\right),$$

where $\psi_0^{(2)}(n)$ is defined as $\Sigma\psi_0(d)$ for all divisors d of n.

Let us further define generally, for any positive integer N,

$$\psi_0^{(N)}(n) \quad \text{as} \quad \Sigma\psi_0^{(N-1)}(d),$$

with, to begin with,

$$\psi_0^{(0)}(n) = 1, \quad \psi_0^{(1)}(n) = \psi_0(n).$$

We obtain by continued repetition of the above process

$$\begin{aligned} F_m(n) &= \Sigma F_{m+4}(d)\,\psi_0^{(3)}\left(\frac{n}{d}\right) \\ &= \Sigma F_{m+5}(d)\,\psi_0^{(4)}\left(\frac{n}{d}\right) \\ &= \dots \\ &= \Sigma F_{m+N}(d)\,\psi_0^{(N-1)}\left(\frac{n}{d}\right). \end{aligned} \tag{E}$$

Here m is still unrestricted. But give it the negative integral value $-N$, and remember that $F(d) = d$, and there results the sum equal to $F_{-N}(n)$ of which we were in search, namely,

$$F_{-N}(n) = \Sigma\, d\psi_0^{(N-1)}\left(\frac{n}{d}\right), \tag{F}$$

a formula of which the early cases are

$$F_{-1}(n) = \Sigma d = \psi_1(n),$$

$$F_{-2}(n) = \Sigma\, d\psi_0\left(\frac{n}{d}\right) = \Sigma \psi_1(d),$$

$$F_{-3}(n) = \Sigma\, d\psi_0^{(2)}\left(\frac{n}{d}\right),$$

&c., &c.

It remains to exhibit $\psi_0^{(N)}(n)$ numerically, for any positive integral n and N. If, expressed as a product of prime factors,

$$n = p_r^\rho p_s^\sigma p_t^\tau \dots,$$

we have $\quad \psi_0^{(1)}(n) = \psi_0(n) = (\rho+1)(\sigma+1)(\tau+1)\dots.$

Consequently $\quad \psi_0^{(2)}(n) = \Sigma\Sigma\Sigma \dots (\rho'+1)(\sigma'+1)(\tau'+1)\dots,$

for the ranges $\quad 0 \leqq \rho' \leqq \rho, \quad 0 \leqq \sigma' \leqq \sigma, \quad 0 \leqq \tau' \leqq \tau,$

$$= \Sigma(\rho'+1)\,\Sigma(\sigma'+1)\,\Sigma(\tau'+1)$$

$$= \frac{(\rho+1)(\rho+2)}{1.2}\,\frac{(\sigma+1)(\sigma+2)}{1.2}\,\frac{(\tau+1)(\tau+2)}{1.2}\dots.$$

Next, in like manner,

$$\psi_0^{(3)}(n) = \frac{(\rho+1)(\rho+2)(\rho+3)}{1.2.3}\,\frac{(\sigma+1)(\sigma+2)(\sigma+3)}{1.2.3}\dots;$$

and generally

$$\psi_0^{(N)}(n) = \frac{(\rho+N)!}{\rho!\,N!}\,\frac{(\sigma+N)!}{\sigma!\,N!}\,\frac{(\tau+N)!}{\tau!\,N!}\dots. \tag{G}$$

The sum (F) is then definitely given as a sum of multiples of divisors d for any n. As a simple example we may write down

$$\begin{aligned} F_{-3}(12) &= 1\psi_0^{(2)}(12)+2\psi_0^{(2)}(6)+3\psi_0^{(2)}(4)+4\psi_0^{(2)}(3)+6\psi_0^{(2)}(2) \\ &\qquad +12\psi_0^{(2)}(1) \\ &= 1\,\frac{3.4}{1.2}\,\frac{2.3}{1.2}+2\left(\frac{2.3}{1.2}\right)^2+3\,\frac{3.4}{1.2}+4\,\frac{2.3}{1.2}+6\,\frac{2.3}{1.2}+12.1 \\ &= 18 \quad + \quad 18 \quad + \ 18 \ + \ 12 \ + \ 18 \ + 12 \\ &= 96, \end{aligned}$$

which is correctly equal to $12\left(1+\frac{3}{2}+\frac{6}{2^2}\right)\left(1+\frac{3}{3}\right)$.

5. We still desire the summation for which $F_N(n)$, for a positive integral N, is the expression.

By (D) we have, for any m,

$$F_m(n) = \Sigma F_{m-1}(d)\,\mu\left(\frac{n}{d}\right),$$

where the definition of $\mu(s)$, for any number s, may be stated that it is the signed unit which is the coefficient of s in the expansion of the product

$$\Pi\,(1-p),$$

for all primes p, if the product equal to s actually occurs in the expansion, and is otherwise zero. Let us further define

$$\mu^{(N)}(s)$$

as the coefficient of s, if it actually occurs, and zero otherwise, in the expansion of the product

$$\Pi\,(1-p)^N.$$

We have, by a repetition of the reduction (D),

$$F_m(n) = \Sigma F_{m-2}(d)\left\{\mu\left(\frac{n}{d_1}\right)\mu\left(\frac{d_1}{d}\right)+\mu\left(\frac{n}{d_2}\right)\mu\left(\frac{d_2}{d}\right)+\dots\right\},$$

for all divisors d of n, where $\frac{n}{d_1}$, $\frac{n}{d_2}$, ... are the divisors of $\frac{n}{d}$. Now the sum in brackets here is the coefficient of the product of primes $\frac{n}{d}$ as it occurs in the product of $\Pi(1-p)$ and $\Pi(1-p)$; in other words, it is $\mu^{(2)}\left(\frac{n}{d}\right)$, the coefficient of $\frac{n}{d}$ as and if it occurs as a product of primes or as 1 in the expansion of the product $\Pi(1-p)^2$. Thus

$$\begin{aligned}
F_m(n) &= \Sigma F_{m-2}(d)\,\mu^{(2)}\left(\frac{n}{d}\right)\\
&= \Sigma F_{m-3}(d)\left\{\mu^{(2)}\left(\frac{n}{d_1}\right)\mu\left(\frac{d_1}{d}\right)+\mu^{(2)}\left(\frac{n}{d_2}\right)\mu\left(\frac{d_2}{d}\right)+\dots\right\}\\
&= \Sigma F_{m-3}(d)\left\{\text{coefficient of } \frac{n}{d} \text{ in product } \Pi(1-p)^2\Pi(1-p)\right\}\\
&= \Sigma F_{m-3}(d)\,\mu^{(3)}\left(\frac{n}{d}\right)\\
&= \dots\\
&= \Sigma F_{m-N}(d)\,\mu^{(N)}\left(\frac{n}{d}\right). \qquad \text{(H)}
\end{aligned}$$

Now give to the unrestricted m the value of the positive integer N. This produces our desired result, namely,

$$F_N(n) = \Sigma d\mu^{(N)}\left(\frac{n}{d}\right). \qquad \text{(K)}$$

It should be noticed that all numbers $F_{-N}(n)$ are positive, but that this is not the case with all numbers $F_N(n)$.

6. A second method of reversing the equality (C) so as to obtain an expression linear in F_{m-1}'s for an F_m, which is at first sight different from (D), is the extension of one given by Glaisher (*Phil. Mag.*, 1884) and Hammond (*Messenger*, 1891) for expressing $\phi(n)$, *i.e.*, $F_1(n)$, linearly in terms of 1, 2, 3, ..., n, *i.e.*, in terms of $F_0(1)$, $F_0(2)$, $F_0(3)$, ..., $F_0(n)$. Write down the n equalities (C) for the values 1, 2, 3, ..., n of n. They furnish n linear equations for the determination of $F_m(1)$, $F_m(2)$, $F_m(3)$, ..., $F_m(n)$. $F_m(1)$ occurs with coefficient 1 in all the equations, $F_m(2)$ with coefficient 1 in the second, fourth, &c., $F_m(3)$ in the third, sixth, &c., and so on, and, lastly, $F_m(n)$ with coefficient 1 in the last only. Thus the determinant of the right-hand sides is unity, and $F_m(n)$ is equal to a

determinant of n^2 constituents, whose first $n-1$ columns—or say rather rows—are

$$\begin{array}{cccccccccc} 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & \ldots, \\ 0, & 1, & 0, & 1, & 0, & 1, & 0, & 1, & 0, & \ldots, \\ 0, & 0, & 1, & 0, & 0, & 1, & 0, & 0, & 1, & \ldots, \\ 0, & 0, & 0, & 1, & 0, & 0, & 0, & 1, & 0, & \ldots, \\ & & & \&c., & & \&c., & & & & \end{array}$$

and whose n-th row is

$$F_{m-1}(1),\ F_{m-1}(2),\ F_{m-1}(3),\ \ldots,\ F_{m-1}(n).$$

By means of (E) it is easy in like manner to write down a determinant expression for $F_m(n)$, with a last row consisting of

$$F_{m-N}(1),\ F_{m-N}(2),\ F_{m-N}(3),\ \ldots,\ F_{m-N}(n),$$

and first $n-1$ rows consisting of $\psi_0^{(N-1)}$s of numbers up to n, with zeroes in places as above.

7. Another easily proved determinant theorem includes the one known as H. J. S. Smith's (*Proc. Lond. Math. Soc.*, Vol. VII., p. 208), that

$$\phi(n)\,\phi(n-1)\,\phi(n-2)\,\ldots\,\phi(1),$$

i.e.,
$$F_1(n)\,F_1(n-1)\,F_1(n-2)\,\ldots\,F_1(1),$$

is equal to the determinant of n^2 constituents in which the constituent in the r-th row and s-th column, for each r and s, is g_{rs}, the G.C.M. of r and s. The more general fact as to our functions is that

$$F_m(n)\,F_m(n-1)\,F_m(n-2)\,\ldots\,F_m(1)$$

is equal to the result of replacing in Smith's determinant each g_{rs} by $F_{m-1}(g_{rs})$.

To prove it take, from (C),

$$F_{m-1}(g_{rn}) = \Sigma F_m(\delta), \text{ for divisors } \delta \text{ of } g_{rn},$$
$$= \Sigma F(d), \text{ for divisors } d \text{ of } n,$$

where $F(d)$ denotes $F_m(d)$ or 0 according as d does or does not divide r. Hence, by the Dedekind-Liouville theorem of reversion,

$$F(n) = \Sigma\mu\left(\frac{n}{d}\right)F_{m-1}(g_{rd}), \text{ for divisors of } n\,;$$

i.e., the right-hand side is equal to $F_m(n)$ or 0 according as n does or

does not divide r. Now take the determinant Δ_n in which the type constituent is $F_{m-1}(g_{rs})$. As

$$\mu\left(\frac{n}{n}\right) = \mu(1) = 1,$$

the value of Δ_n is not altered when we replace the n-th column by the sum of the multiples $\mu\left(\frac{n}{d}\right)$ of the various d-th columns, for divisors d of n. The last constituent in the column thus becomes $F_m(n)$, as n divides n, and the r-th constituent for any $r<n$ becomes 0, as n does not divide r. Thus

$$\Delta_n = F_m(n)\,\Delta_{n-1},$$

where the formation of Δ_{n-1}, of $(n-1)^2$ constituents, is according to the same law as that of Δ_n. Repeating this argument $n-1$ times, and noticing that

$$\Delta_1 = F_{m-1}(1) = 1 = F_m(1).$$

we have, as stated,

$$\Delta_n = F_m(n)\,F_m(n-1)\,F_m(n-2)\,\ldots\,F_m(1).$$

The proof is of course applicable when instead of F_m, F_{m-1} we have any two arithmetical functions χ, λ which are such that for every n

$$\lambda(n) = \Sigma\chi(d).$$

Probably the theorem in its generality ought to be regarded as known. A generalization of his theorem given by Smith himself (*loc. cit.*) produces it with the aid of Dedekind's reversion.*

8. Let us examine more closely the linear expression for an F_m in terms of F_{m-1}'s which is exhibited in determinant form in § 6. Though in form very unlike the expression (D), it may be seen to be really equivalent to it. As far as I know it has not been noticed even that the Glaisher-Hammond determinant expression for $\phi(n)$ is really the same as the expression $\Sigma\mu\left(\frac{n}{d}\right)d$. The statement of this is, in accordance with what we have seen, a case of the more general statement as to the expressions for an F_m in terms of F_{m-1}'s; and this again is a class of cases of the more general statement that, if we have two arithmetical functions $\lambda(n)$, $\chi(n)$ such that for every n

$$\lambda(n) = \Sigma\chi(d),$$

* [The theorem has been given explicitly by Cesaro, *Nouvelles Annales* (3), v., p. 44.]

the two apparently different reversions of this

$$\chi(n) = \Sigma\mu\left(\frac{n}{d}\right)\lambda(d),$$

$$\chi(n) = \begin{vmatrix} 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & \dots, & 1 \\ 0, & 1, & 0, & 1, & 0, & 1, & 0, & 1, & \dots & \\ 0, & 0, & 1, & 0, & 0, & 1, & 0, & 0, & \dots & \\ 0, & 0, & 0, & 1, & 0, & 0, & 0, & 1, & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & & \dots \\ \lambda(1), & \lambda(2), & \lambda(3), & \lambda(4), & \lambda(5), & \lambda(6), & \lambda(7), & \lambda(8), & \dots, & \lambda(n) \end{vmatrix}$$

are in effect the same, the functions $\lambda(r)$ for r's which do not divide n being mere superfluities in the last row.

We will prove that $\Sigma\mu\left(\frac{n}{d}\right)\lambda(d)$ is a factor of the determinant, and that the other factor is unity. This will be shown, since

$$\mu(1) = 1,$$

if we can prove that, by adding to the last column multiples $\mu\left(\frac{n}{d}\right)$ respectively of the other d-th columns, we can reduce all the constituents but the last in the n-th column to zero. For, except in the last row, all constituents below and to the left of the principal diagonal vanish, and those in the principal diagonal are units.

Let ϕ_{rs} denote 0 or 1 according as r and s are unequal or equal. The constituent C_{rs} in the r-th row and s-th column ($r<n$) is 0 or 1 according as s is not or is a multiple of r. Thus

$$C_{rs} = \Sigma\phi_{r\delta}, \text{ for divisors } \delta \text{ of } s,$$

or, with changed notation, $C_{rn} = \Sigma\phi_{rd}$.

Now this necessitates that

$$\phi_{rn} = \Sigma\mu\left(\frac{n}{d}\right)C_{rd},$$

and consequently that the right-hand sum vanishes except for $r=n$. This proves what was required. It has incidentally found the values —some zero, some 1, and some -1—of the determinants of $n-1$ rows and columns which are obtained by omitting single columns from the first $n-1$ rows of the determinant $\chi(n)$.

Of course the whole determinant need not be written as one of n, but only as one of $\psi_0(n)$, rows and columns. Of the numbers

$1, 2, 3, \ldots, n$ let $1, d_2, d_3, \ldots, n$ be those, in order, which are divisors of n, and $e_1, e_2, e_3, \ldots$ those which are not. The numbers of d's and e's are $\psi_0(n)$ and $n-\psi_0(n)$ respectively. In the various e-th rows the constituents in the various d-th columns are all zero—for no e divides any d. Accordingly the determinant is the product of two determinants of orders $\psi_0(n)$ and $n-\psi_0(n)$, the former containing the constituents at intersections of the various d-th rows and the various d-th columns, and the latter those at intersections of e-th rows and e-th columns. The latter determinant is equal to unity—for the constituents in its principal diagonal are units, while those below and to the left of that diagonal are all zero. The former is a determinant of order $\psi_0(n)$, which has for its last row

$$\lambda(1),\ \lambda(d_2),\ \lambda(d_3),\ \ldots,\ \lambda(n).$$

What the multipliers of these are, in the expansion of the determinant, has been completely seen above.

With regard to the particular application to arithmetical functions F_m, F_{m-1}, we are then assured that the Glaisher-Hammond determinant method adds no essentially new information to that afforded by the algebraical identities which we dealt with at the outset.

9. A known fact of some generality (*cf.* Bachmann, *Encyclopädie*, Band I., p. 650) is that, if $\lambda_1, \chi_1, \lambda_2, \chi_2$ are four arithmetical functions such that, for all numbers n.

$$\lambda_1(n) = \Sigma\chi_1(d) \quad \text{and} \quad \lambda_2(n) = \Sigma\chi_2(d),$$

then

$$\Sigma\chi_1\left(\frac{n}{d}\right)\lambda_2(d) = \Sigma\chi_2\left(\frac{n}{d}\right)\lambda_1(d).*$$

This has an interesting application to our functions F_m. Since

$$F_{m-1}(n) = \Sigma F_m(d) \quad \text{and} \quad F_{\mu-1}(n) = \Sigma F_\mu(d),$$

it gives that

$$\Sigma F_m\left(\frac{n}{d}\right)F_{\mu-1}(d) = \Sigma F_\mu\left(\frac{n}{d}\right)F_{m-1}(d).$$

---

* A very convenient way of rendering visible such transformations of summations is to arrange elements in a square array and equate their sum taken by rows to their sum taken by columns. Thus, if we take a row and a column for each divisor of n, and place, in each d row, $\chi_1(d)\,\chi_2\left(\frac{n}{\delta}\right)$ or 0 in the δ-column according as d is or is not a divisor of δ, the two different ways of adding elements give the equality here before us. Most familiar arithmetical equalities can be made clear in this sort of way.

In particular $\Sigma F_m\left(\frac{n}{d}\right)F_{-m}(d) = \Sigma F_{1-m}\left(\frac{n}{d}\right)F_{m-1}(d)$

$$= \Sigma F_{m-1}\left(\frac{n}{d}\right)F_{1-m}(d).$$

Taking m integral in this, we get

$$\Sigma F_N\left(\frac{n}{d}\right)F_{-N}(d) = \Sigma F_{N-1}\left(\frac{n}{d}\right)F_{1-N}(d) = \Sigma F_{N-2}\left(\frac{n}{d}\right)F_{2-N}(d)$$

$$= \dots$$

$$= \Sigma F_0\left(\frac{n}{d}\right)F_0(d)$$

$$= \Sigma\left(\frac{n}{d}d\right) = n\psi_0(n).$$

So, too,
$$\Sigma F_{N+1}\left(\frac{n}{d}\right)F_{-N}(d) = \Sigma F_1\left(\frac{n}{d}\right)F_0(d)$$

$$= \Sigma d\phi\left(\frac{n}{d}\right) = n\Sigma\frac{\phi(d)}{d};$$

$$\Sigma F_{N-1}\left(\frac{n}{d}\right)F_{-N}(d) = \Sigma F_0\left(\frac{n}{d}\right)F_{-1}(d)$$

$$= \Sigma\frac{n}{d}\psi_1(d) = n\Sigma\frac{\psi_1(d)}{d};$$

and more generally

$$\Sigma F_{N+r}\left(\frac{n}{d}\right)F_{-N}(d) = n\Sigma\frac{F_r(d)}{d},$$

for any positive or negative r.

These equalities can be at once stated in other terms by means of (F) and (K).

10. In conclusion the remark may be made that examples may with ease be written down of other identities than those in §§ 1, 2 which yield expressions for simple arithmetical sums. For instance, take

$$p_r^{2\rho}p_s^{2\sigma}p_t^{2\tau}\dots\Pi\left(1-\frac{1}{p^2}\right)^{-1} \equiv \Sigma\left(p_r^{2\rho'}p_s^{2\sigma'}p_t^{2\tau'}\dots\right) + \text{fractional terms},$$

where $\quad 0 \leqq \rho' \leqq \rho, \quad 0 \leqq \sigma' \leqq \sigma, \quad 0 \leqq \tau' \leqq \tau,$ &c.

The non-fractional part on the right is the sum of the squares of the algebraic divisors of $p_r^\rho p_s^\sigma p_t^\tau \ldots$. Upon multiplication by $\Pi\left(1-\frac{1}{p}\right)$ this yields an identity which gives

$$\text{integral part of expansion of } p_r^{2\rho} p_s^{2\sigma} p_t^{2\tau} \ldots \Pi\left(1+\frac{1}{p}\right)^{-1}$$

$$\equiv \Sigma\left\{\text{integral part of expansion of } p_r^{2\rho'} p_s^{2\sigma'} p_t^{2\tau'} \ldots \Pi\left(1-\frac{1}{p}\right)\right\}.$$

Now take the prime numbers for the p's, and any number

$$n = p_r^\rho p_s^\sigma p_t^\tau \ldots,$$

thus getting

part of $n^2\Pi\left(1+\frac{1}{p}\right)^{-1}$ integral in p's, when n is expressed as a product of p's,

$$= \Sigma\phi\,(d^2), \text{ for all divisors } d \text{ of } n\,;$$

a result which may also be written

$$S-S' = \Sigma\phi\,(d^2) = \Sigma d\phi\,(d) = 2\Sigma\phi_1\,(d)-1,$$

where $\phi_1\,(d)$ denotes the sum of the numbers prime to and not exceeding d, and S, S' denote the sums of the divisors of n^2 which are and are not respectively products of even (including zero) numbers of unequal or equal prime factors.

In like manner the integral parts, when n is replaced by its expression as a product of primes, of the expansions of

$$n^3\Pi\left(1+\frac{1}{p}+\frac{1}{p^2}\right)^{-1},$$

$$n^4\Pi\left(1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}\right)^{-1},$$

&c., &c.,

are

$$\Sigma\phi\,(d^3) = \Sigma\, d^2\phi\,(d),$$

$$\Sigma\phi\,(d^4) = \Sigma\, d^3\phi\,(d),$$

&c., &c.

The Elementary Theory of Cauchy's Principal Values. By G. H. Hardy. Received March 12th, 1901. Read March 14th, 1901.

1. Definite integrals are of two kinds—*finite* and *infinite*.* In finite integrals the range of integration is finite, and the subject of integration finite throughout it. A finite integral, simple or multiple, is defined as a *single limit;* thus, for instance, the simple integral

$$\int_a^A f(x)\,dx \tag{1}$$

is the limit when n tends to infinity of a certain finite sum

$$\sum_{r=1}^{n} f(\xi_{r-1,r})(x_r - x_{r-1}),$$

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = A, \quad x_{r-1} \leqq \xi_{r-1,r} \leqq x_r.$$

When a or A is infinite, or $f(x)$ has infinities lying within (a, A), the integral (1) can be defined, if at all, only as a double limit. Thus, if, to take the simplest case, $f(x)$ has a single infinity ξ in (a, A), it must be defined as the limit of

$$\left(\int_a^{\xi-\epsilon} + \int_{\xi+\epsilon'}^{A}\right) f(x)\,dx,$$

when the positive quantities ϵ, ϵ' tend, independently or otherwise, to zero; that is to say, as the limit of the sum of two single limits, *i.e.*, a double limit. The only case which is considered in any detail in the books is that in which this limit is determinate when ϵ, ϵ' tend independently to zero; that is, when it is the same for all possible ways in which they can do so. We shall say then that the integral (1) is *unconditionally* convergent. Unconditional convergence may be of two kinds—*absolute* or *relative;* but this is a distinction with which we need not concern ourselves at present.

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* There are, so far as I know, no English words of general use in this connexion equivalent to the German *eigentlich, uneigentlich*. "Finite" and "infinite" do, I think, really express the distinction in a way the German words do not. It has indeed been suggested to me as a possible objection that "infinite integral" ought to mean "integral whose value is infinite, divergent integral." But nobody supposes that an "infinite series" is necessarily divergent, and I hardly see why confusion should be more likely to arise in one case than in the other.

Conditionally Convergent Integrals.

2. We shall now suppose that this is not the case. We have then to consider the possibility that a definite limit may result if the quantities ϵ, ϵ', while they tend to zero, continue to satisfy one or more relations. If such relations can be found, we shall say that the function under the integral sign is *conditionally integrable*; that the integral is *conditionally convergent*; and that the limit which corresponds to any such particular set of relations is a *particular value* of the integral.

These definitions can only be useful when the subject of integration changes its sign within the range; the integral of a function of constant sign is either unconditionally (and absolutely) convergent or determinately divergent. And when a function is only conditionally integrable different sets of relations will generally lead to different results.

3. Consider, for example, the simple case of a function $f(x)$ which is finite and integrable throughout the range (a, A) except at one point X; and is positive throughout a finite interval $(X-\xi, X)$, negative throughout a finite interval $(X, X+\xi')$. Suppose, moreover, that

$$\lim_{\epsilon=0}\int_a^{X-\epsilon} = +\infty, \quad \lim_{\epsilon'=0}\int_{X+\epsilon'}^{A} = -\infty.$$

Then *any quantity m whatever is a particular value of*

$$\int_a^A f\,dx.$$

For let $\eta_1, \eta_2, \ldots$ and $\eta'_1, \eta'_2, \ldots$ be any two sequences of descending positive quantities whose limits are zero. Let

$$\int_a^{\xi-\eta_i} = H_i, \quad \int_{X+\eta'_i}^{A} = -H'_i.$$

Let $m_1, m_2, \ldots$ be any sequence of quantities whose limit is m. We can determine M_1, M'_1 so that $M_1 > H_1$, $M'_1 > H'_1$, and $M_1 - M'_1 = m_1$. Then we can determine $M_2 > M_1$, $M'_2 > M'_1$ so that $M_2 > H_2$, $M'_2 > H'_2$, and $M_2 - M'_2 = m_2$; and so on.

We can then determine ϵ_i, ϵ'_i by the equations

$$\int_a^{X-\epsilon_i} = M_i, \quad \int_{X+\epsilon'_i}^{A} = -M'_i;$$

so that
$$\int_a^{X-\epsilon_i} + \int_{X+\epsilon_i'}^A = m_i.$$

Also $\epsilon_i < \eta_i$, $\epsilon_i' < \eta_i'$; so that $\lim \epsilon_i = 0$, $\lim \epsilon_i' = 0$. Hence, if ϵ tend to 0 through the sequence of values $\epsilon_1, \epsilon_2, \ldots$, and ϵ' through the sequence $\epsilon_1', \epsilon_2', \ldots$,
$$\lim_{\epsilon, \epsilon' = 0} \left(\int_a^{X-\epsilon} + \int_{X+\epsilon'}^A \right) = m.$$

Thus m is a particular value of $\int_a^A$. For instance
$$\lim_{\epsilon = \kappa\epsilon' = 0} \left(\int_a^{X-\epsilon} + \int_{X+\epsilon'}^A \right) \frac{dx}{x-X} = \log \left(\kappa \frac{A-X}{X-a} \right),$$

which may be made equal to any quantity we please by choice of κ.

4. There is only one form of "particular value" with which we shall be concerned in the following pages.

The Principal Value of an Integral.

Suppose that $f(x)$ possesses a convergent integral over any part of (a, A) which does not include any of a finite number n of points X_i, distinct from a or A, and that
$$\left(\int_a^{X_1-\epsilon_1} + \int_{X_1+\epsilon_1}^{X_2-\epsilon_2} \ldots + \int_{X_n+\epsilon_n}^A \right) f(x)\, dx$$

tends to a finite limit when the quantities $\epsilon_1, \ldots, \epsilon_n$ tend independently to 0. Then this limit will be called *the principal value of the integral* $\int_a^A$, and will be denoted by
$$P \int_a^A f(x)\, dx.$$

5. *Historical and Critical Note.*—The set of relations which serves to define the principal value is, in fact,
$$\epsilon_i = \epsilon_i' \quad (i = 1, 2, \ldots, n).$$

The principal value was first defined by Cauchy. But Cauchy's ideas on the subject of infinite integrals had not the degree of precision required by modern analysts. So far as I am aware, he does not recognize the distinction between unconditionally and conditionally convergent infinite integrals at all. In some of his earlier memoirs, indeed, he does not distinguish principal values from ordinary finite integrals. And he does not seem to have observed that, if the subject of

integration becomes infinite like $(x-X)^{\mu}$, the only case in which the definition of the principal value is useful is that of $\mu = 1$.

There is so much that seems arbitrary, at any rate from the point of view of the theory of functions of a real variable, about the conditions

$$\epsilon_i = \epsilon_i'$$

by which the principal value is defined, that its theory has been practically neglected. Thus Riemann, who was the first to give a precise form to the definition of the infinite integral, expressly excludes it from consideration. And in the best theoretical treatises (as, *e.g.*, Stolz, Jordan, Harnack) it is generally dismissed with a remark; sometimes its very legitimacy appears to be called in question. For instance:

"Cauchy hat . . . Hauptwerthe (*valeurs principales*) in Betracht gezogen, auch wenn . . . keinen Sinn hat. *Es ist jedoch uns unseren Ueberlegungen klar, dass man besser thut, dieser Einfuhrung nicht zu folgen*" (Kronecker, *Vorlesungen*, p. 211).

"Dass der Begriff der *valeur principale* einen Integrales, den Cauchy aufstellt. nicht statthaft sei, braucht nicht erörtert zu werden" (Schläfli, *Acta Math.*, Vol. VII., p. 187).

It is, at any rate, quite clear that the principal value is not what the last writer asserts it to be: "was er [Cauchy] so nennt, ist *eine Summe von Integralen, die einander nichts angehn.*" For instance, if $f(x)$ be a function of the complex variable, analytic near the origin,

$$P\int_{-1}^{1} \frac{f(x)}{x}\,dx = \lim_{\epsilon=0} \left(\int_{\epsilon}^{1} + \int_{-1}^{-\epsilon}\right)$$

is determinate. But the principal value is neither the sum of $\int_{\epsilon}^{1}$, $\int_{-1}^{-\epsilon}$ nor the sum of $\int_{0}^{1}$, $\int_{-1}^{0}$, which are not convergent.

Again, in the *Encyc. d. Math. Wiss.* (I. ii. 1, p. 38) it is asserted that the ordinary formula of transformation cannot be applied to principal values. We shall see later that in such cases as ordinarily occur this statement is untrue. The fact is that the interest of the principal value depends upon its frequent occurrence in connexion with the ordinary, elementary functions of analysis, such as

$$\frac{1}{x}, \quad \log x, \quad \operatorname{cosec} x.$$

These are of course extremely *special* functions. But we must distinguish, with Borel ("Mémoire sur les Séries divergentes," *Ann. de l'E. N.*, XVI.), between the *theoretically general* and the *practically general*. The simplest special kind of infinity of a function across which its integral is only conditionally convergent is a *simple pole*; and in general analysis this is, of course, the most important kind of all. Indeed, when we look at the matter from the standpoint of the theory of functions of a complex variable, and consider the methods used in contour integration, in which poles are often excluded from the range of integration by semicircles having the poles as centres, the particular significance of the at first sight arbitrary conditions $\epsilon_i = \epsilon_i'$ becomes quite plain.

It is worth remarking that formulæ involving principal values are very often

simpler than the corresponding formulæ which involve ordinary integrals. Thus, for instance,

$$\int_0^\infty \frac{dx}{a^2+x^2} = \frac{\pi}{2a}, \quad P\int_0^\infty \frac{dx}{a^2-x^2} = 0;$$

$$\int_0^\pi \frac{dx}{\cosh\alpha - \cos x} = \frac{\pi}{\sinh\alpha}, \quad P\int_0^\pi \frac{dx}{\cos\alpha - \cos x} = 0.$$

The consequence of this is that the easiest way of evaluating an ordinary integral is often by means of its connexion with a principal value. And by the use of principal values the range of some of the fundamental double limit problems of the integral calculus can be considerably extended. I hope to illustrate these points systematically in a series of further papers. For the present I may refer to a paper in the *Quart. Jour. of Math.* (June and September, 1900).

Elementary Properties.

6. (i.) $$P\int_a^A f(x)\,dx = -P\int_A^a f(x)\,dx.$$

(ii.) $$P\int_a^A \sum_1^n f_i(x)\,dx = \sum_1^n P\int_a^A f_i(x)\,dx.$$

(iii.) $$P\int_a^A \kappa f(x)\,dx = \kappa P\int_a^A f(x)\,dx.$$

(iv.) *If* $P\int_a^A f(x)\,dx$ *is determinate, so is* $P\int_a^c f(x)\,dx$ $(a<c<A)$, *except possibly for a finite number of values of* c, *and*

$$P\int_a^c + P\int_c^A = P\int_a^A.$$

(v.) $P\int_a^A f(x)\,dx$ *is a continuous function of* A *for all values of* A *for which it is defined. It has a derivate equal to* $f(A)$ *for all values of* A *for which* $f(A)$ *is continuous.*

We need not delay to prove the above almost obvious theorems, which result immediately from the definition and the corresponding properties of ordinary integrals. We will only remark that they are equally true of any particular value of an integral $\int_a^A$, when the range only includes a finite number of points across which the integral is not unconditionally convergent.

(vi.) *If $f(x)$ be continuous in general in (a, A), and $F(x)$ be continuous except at a finite number of points X_i, distinct from a or A, and there become infinite or discontinuous in such a way that*

$$\lim_{\epsilon=0} \{f(X_i-\epsilon)-f(X_i+\epsilon)\} = 0,$$

and if $F(x)$ have a derivate equal to $f(x)$ at all points at which the latter is continuous, then, for all values of x in (a, A) other than X_i,

$$F(x)-F(a) = P\int_a^x f(x)\,dx.$$

For, if, *e.g.*, $X_i < x < X_{i+1}$,

$$\int_{X_i+\epsilon_i}^{x} f(x)\,dx = F(x)-F(X_i+\epsilon_i),$$

$$\int_{X_{i-1}+\epsilon_{i-1}}^{X_i-\epsilon_i} f(x)\,dx = F(X_i-\epsilon_i)-F(X_{i-1}+\epsilon_{i-1}),$$

$$\dots \qquad \dots \qquad \dots$$

and on adding and proceeding to the limit the theorem follows.

7. Suppose, in particular, that $\psi(x)\,\phi(x)$ is a product of two functions which satisfies the conditions imposed upon $F(x)$, while $\psi(x)\,\phi'(x)$, $\psi'(x)\,\phi(x)$ satisfy those imposed upon $f(x)$. Then

$$P\int_a^A \{\psi(x)\,\phi'(x)+\psi'(x)\,\phi(x)\}\,dx = \Big[\phi(x)\,\psi(x)\Big]_a^A,$$

the formula for *integration by parts*.

Let, *e.g.*, $a = 0$, $A = \infty$, $\psi(x) = x$, $\phi(x) = \log\left(1-\frac{p}{x^2}\right)^2$,

where $p > 0$. Then

$$\int_0^\infty \log\left(1-\frac{p}{x^2}\right)^2 dx = 4P\int_0^\infty \frac{p\,dx}{p-x^2} = 0.$$

We only defined $P\int_a^A$ when A was finite. But, if there be only a finite number of points X_i, all $< H$, $P\int_a^\infty$ is simply $P\int_a^H + \int_H^\infty$, if this be determinate.

Convergence of the Principal Value over an Isolated Infinity.*

7. The principal value $P\int_a^A$ has so far only been defined in the case in which (a, A) includes but a finite number of points X_i across which the integral is not unconditionally convergent. Wider definitions will be given shortly. But first we must consider more in detail the possible characters of these points X_i. There is only one case with which we need seriously concern ourselves. As we have already pointed out, the principal value is a special notion which derives its interest from its frequent occurrence in connexion with certain familiar functions. It would therefore be futile to attempt to state theorems connected with it with the utmost generality of which they are capable. That would be to try to generalize what is essentially a special case. Our object will be rather to prove a few theorems general enough to give an account of such cases as we shall meet.

Infinities X^r.

8. We define the functions

$$lx,\ l^2x,\ \dots$$

by the equations $\quad lx = \log|x|, \quad l^2x = llx, \ \dots .$

We shall say that a function $f(x)$ has an infinity X^r at a point $x = X$, if a finite interval $(X-\xi, X+\xi)$ can be found within which $f(x)$ can be expressed in the form

$$\psi_\nu(x-X)\,\Theta(x),$$

where (i.) $\Theta(x)$ is a function which possesses a continuous derivate throughout $(X-\xi, X+\xi)$, and

$$\text{(ii.)} \quad \psi_\nu(u) \equiv |u|^{-r}\,|lu|^{r_1}\,|l^2u|^{r_2}\dots|l^\nu u|^{r_\nu}.$$

* It is, no doubt, verbally inaccurate to say that "the principal value is convergent"; the principal value is a limit to which something else converges. However the expression saves a good many rather awkward periphrases. And it is quite usual to say that an ordinary infinite integral converges, although this is open to the same criticism. Strictly speaking, to say that

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

is convergent is only a short way of saying that it is a limit to which something else converges.

It is to be understood that some or all of the symbols of the absolute value on the right may be omitted, provided no difficulty as to reality arises. Thus $\psi_\nu(u)$ might be

$$u^{\frac{1}{2}}, \quad |u|^{-\frac{1}{2}}, \quad u^{\frac{1}{2}}\,|lu|^{-\frac{1}{2}}, \ \ldots .$$

It is also to be observed that, if $r < 0$, or if $r = r_1 \ldots = r_{i-1} = 0$, $r_i < 0$, the point is really not an infinity, but a zero. But no confusion will arise from this.

9. To avoid misunderstanding later we add the following remarks on the subject of these logarithmic factors. All of

$$lu, \quad l^2u, \quad l^3u, \quad \ldots$$

become infinite for $u = 0,\ \infty$. But also

$$\left.\begin{aligned} |l^2u| &= \infty, \quad u = \pm 1 \\ &= 0, \quad\ \ u = \pm e, \quad \pm e^{-1} \end{aligned}\right\},$$

$$\left.\begin{aligned} |l^3u| &= \infty, \quad u = \pm 1, \quad \pm e, \quad \pm e^{-1} \\ &= 0, \quad\ \ u = \pm e^e, \quad \pm e^{-e}, \quad \pm e^{e^{-1}}, \quad \pm e^{-e^{-1}} \end{aligned}\right\},$$

and so on. We are, however, interested in these logarithmic products only in connexion with the behaviour of a function *in the immediate neighbourhood of* $u = 0$. So we shall suppose all these other possible infinities excluded from consideration, either by a sufficient restriction of the range of integration, or by a suitable choice of exponents $r, r_1, \ldots, r_\nu$.

Since, if $\phi(u)$ be real,

$$\frac{d}{du}\log|\phi(u)| = \frac{|\phi(u)|}{\phi(u)}\,\frac{\phi'(u)}{|\phi(u)|} = \frac{\phi'(u)}{\phi(u)},$$

$$\frac{d}{du}\,l^\nu u = \frac{1}{u\,lu\,l^2u \ldots l^{\nu-1}u};$$

just as

$$\frac{d}{du}\log^\nu u = \frac{1}{u\log u\log^2 u \ldots \log^{\nu-1}u}.$$

As soon as u is small enough $\qquad l^\nu u = \log^\nu u$;

but, by using the functions l^ν, a good deal of possible ambiguity as to sign and reality is avoided.

It is well known that *the integral* $\int f(x)\,dx$ *is absolutely convergent across* X, *if* $r < 1$, *or if* $r = 1$, $r_1 < -1$, *or if* $r = 1$, $r_1 = r_2 \ldots r_{i-1} = -1$, $r_i < -1$. When $r = 1$, we shall write Ω_ν for ψ_ν; so that

$$\Omega_\nu(u) = \frac{1}{u}\prod_1^\nu |l^iu|^{r_i}.$$

Some of the symbols of the absolute value may possibly be omitted.

[redacted]

In what follows we shall suppose, for simplicity, as we evidently may, that the range is so restricted that all of the functions $l^i u$, $i = 1, \ldots, \nu$ are positive.

THEOREM.—*If* $$f(x) = \Omega_\nu (x-X)\,\Theta(x),$$
where $\Theta(x)$ *is a function which has a continuous derivate near* $x = X$, *and*
$$\Omega_\nu(u) = \frac{1}{u} \prod_1^\nu \{l^i u\}^{r_i};$$
and ξ *be a sufficiently small finite quantity, then*
$$P\int_{X-\xi}^{X+\xi} f(x)\,dx$$
will be convergent.

For $$\left(\int_{X+\epsilon}^{X+\xi} + \int_{X-\xi}^{X-\epsilon}\right) f(x)\,dx$$
$$= \int_\epsilon^\xi \{f(X+u)+f(X-u)\}\,du$$
$$= \int_\epsilon^\xi \Omega_\nu(u)\,\{\Theta(X+u)-\Theta(X-u)\}\,du$$
$$= 2\int_\epsilon^\xi \prod_1^\nu \{l^i u\}^{r_i}\,\Theta'(X+\theta u)\,du, \quad (-1 \leqq \theta \leqq 1);$$
and the limit of this for $\epsilon = 0$ is plainly finite and determinate, since
$$\lim_{u=0} u^\gamma \prod_1^\nu \{l^i u\}^{r_i} = 0,$$
for any positive value of γ. Hence the theorem follows.

It should be observed (i.) that, if the exponents r_i satisfy certain conditions, not only $P\int_{X-\xi}^{X+\xi}$, but $\int_{X-\xi}^{X+\xi}$ also, is convergent; (ii.) that $\int (x-X)\,f(x)\,dx$ is evidently convergent across $x = X$; and (iii.) that $\int^x f(x)\,dx$ becomes at most logarithmically infinite for $x = X$, in the cases in which it does not converge up to $x = X$.

That is to say, as $\phi(x) = \int^x f(x)\,dx$ becomes infinite, its modulus remains less than the value of a certain logarithmic product. This may be proved as follows:—

In the first place

$$\phi(X-\epsilon) = \int_\epsilon f(X-u)\,du$$

$$= -\int_\epsilon \frac{1}{u}\prod_1^\nu \{l^i u\}^{r_i}\,\Theta(X-u)\,du.$$

We may suppose the upper limit (say ξ) so small that Θ is of constant sign (say >0), and

$$l^i u > 1,\ i = 1,\ \ldots,\ \nu \quad (0 < u \leqq \xi). \tag{a}$$

Also we may suppose $r_2, r_3, \ldots, r_\nu$ all > 0, and $r_1+1 > r_2+r_3\ldots+r_\nu$. For, if these conditions are not satisfied by the indices r_i, we can substitute for f a function f' whose indices r_i' satisfy the conditions

$$r_2', r_3', \ldots, r_\nu' > 0, \quad r_i' \geqq r_i, \quad r_1'+1 > r_2'+r_3'\ldots+r_\nu';$$

and then, if we can prove our conclusion for f', it will follow *a fortiori* for f, in virtue of (a). Then

$$\phi(X-\epsilon) = -\frac{1}{r_1+1}\int_\epsilon^\xi \frac{d}{du}\{lu\}^{r_1+1}\prod_2^\nu\{l^i u\}^{r_i}\,\Theta(X-u)\,du$$

$$= -\frac{1}{r_1+1}\left[\{lu\}^{r_1+1}\prod_2^\nu\{l^i u\}^{r_i}\,\Theta\right]_\epsilon^\xi - \frac{1}{r_1+1}\int^\xi \{lu\}^{r_1+1}\prod_2^\nu\{l^i u\}^{r_i}\,\Theta'\,du$$

$$+\frac{1}{r_1+1}\sum_2^\nu r_i\int_\epsilon^\xi \frac{1}{u}\{lu\}^{r_1}\prod_2^i\{l^\kappa u\}^{r_\kappa-1}\prod_{i+1}^\nu\{l^\kappa u\}^{r_\kappa}\,\Theta\,du$$

$$= \Phi(\epsilon) + \frac{1}{r_1+1}\sum_2^\nu r_i\psi_i(\epsilon), \text{ say.}$$

Here $\Phi(\epsilon)$ is the sum of

$$\frac{1}{r_1+1}\{l\epsilon\}^{r_1+1}\prod_2^\nu\{l^i\epsilon\}^{r_i}\,\Theta(X-\epsilon)$$

and terms which remain finite as ϵ tends to zero. And we may therefore suppose ξ so small that

$$\Phi(\epsilon) > 0, \quad 0 < \epsilon \leqq \xi.$$

Also $\psi_i(\epsilon) > 0$, and, since the subject of integration in ψ_i is less than that in ϕ, for every value of u in question [by (a)]

$$\phi(X-\epsilon) > \psi_i(\epsilon) \quad (i = 2, 3, \ldots, \nu).$$

Hence
$$0 < \left(1-\frac{\sum_2^\nu r_i}{r_1+1}\right)\phi(X-\epsilon) < \phi(X-\epsilon) - \frac{1}{r_1+1}\sum_2^\nu r_i\psi_i(\epsilon) = \Phi(\epsilon);$$

i.e.,
$$\phi(X-\epsilon) < \frac{r_1+1}{r_1+1-\sum_2^\nu r_i}\,\Phi(\epsilon) < \frac{1}{r_1+1-\sum_2^\nu r_i}\left[H\{l\epsilon\}^{r_1+1}\prod_2^\nu\{l^i\epsilon\}^{r_i}+K\right],$$

where H, K are some finite constants.

Thus our conclusion is established.

11. We shall not have occasion to consider in any detail principal values other than those in which the subject of integration behaves like this near each isolated

infinity across which the integral ceases to be unconditionally convergent. An example of another kind is

$$P\int_{-1}^{1}\sin\frac{1}{x}\,\frac{dx}{x^2}=\lim_{\epsilon=0}\left\{\cos\frac{1}{\epsilon}-\cos\frac{1}{\epsilon}\right\}=0.$$

12. We shall also find the following lemma useful in the sequel.

A Lemma analogous to the First Theorem of the Mean.

LEMMA.—*If* $\Theta(x)$, $\phi(x)$ *be functions which have continuous derivates in* $(X-\xi,\ X+\xi)$, *and* $\Theta(x)$ *do not change its sign, and*

$$f(x)=\Omega_\nu(x-X)\,\Theta(x),$$

then will

$$P\int_{X-\xi}^{X+\xi}f(x)\,\phi(x)\,dx$$

$$=\phi(X)\,P\int_{X-\xi}^{X+\xi}f(x)\,dx+\phi'(X+\mu)\int_{X-\xi}^{X+\xi}(x-X)f(x)\,dx,$$

where $$-\xi\leqq\mu\leqq\xi.$$

That each of these three terms is determinate follows immediately from what precedes. Also

$$\left(\int_{X-\xi}^{X-\epsilon}+\int_{X+\epsilon}^{X+\xi}\right)f\phi\,dx$$

$$=\int_{\epsilon}^{\xi}\{f(X+u)\,\phi(X+u)+f(X-u)\,\phi(X-u)\}\,du$$

$$=\phi(X)\int_{\epsilon}^{\xi}\{f(X+u)+f(X-u)\}\,du$$

$$+\int_{\epsilon}^{\xi}f(X+u)\,\{\phi(X+u)-\phi(X)\}\,du$$

$$-\int_{\epsilon}^{\xi}f(X-u)\,\{\phi(X)-\phi(X-u)\}\,du.$$

When ϵ tends to zero the first term on the right tends to

$$\phi(X)\,P\int_{X-\xi}^{X+\xi}f(x)\,dx.$$

The second is

$$\int_{\epsilon}^{\xi}f(X+u)\left[u\phi'(X+\theta u)\right]du\quad(0\leqq\theta\leqq1)$$

$$=\phi'(X+\lambda)\int_{\epsilon}^{\xi}uf(X+u)\,du\quad(0<\lambda\leqq\xi).$$

The first form shows that the term tends to a limit for $\epsilon = 0$. The second factor of the second form also tends to a limit; and therefore $\phi'(X+\lambda)$ tends to a limit, which must evidently be

$$\phi'(X+\kappa) \quad (0 \leqq \kappa \leqq \xi).$$

Thus the limit of the second term is

$$\phi'(X+\kappa)\int_0^{\xi} uf(X+u)\,du = \phi'(X+\kappa)\int_X^{X+\xi}(x-X)f(x)\,dx.$$

Similarly the third term tends to

$$\phi'(X-\kappa')\int_{X-\xi}^{X}(x-X)f(x)\,dx.$$

Since $(x-X)f(x)$ is of constant sign, we may replace the sum of these two limits by

$$\phi'(X+\mu)\int_{X-\xi}^{X+\xi}(x-X)f(x)\,dx,$$

and the lemma is proved.

If, in particular, $\Theta \equiv 1$,

$$P\int_{X-\xi}^{X+\xi}\Omega_\nu(x-X)\,\phi(x)\,dx = \phi'(X+\mu)\int_{X-\xi}^{X+\xi}(x-X)\,\Omega_\nu(x-X)\,dx.$$

Thus, *e.g.*,
$$P\int_{X-\xi}^{X+\xi}\frac{\phi(x)\,dx}{x-X} = 2\xi\phi'(X+\mu) \quad (-\xi \leqq \mu \leqq \xi).$$

Infinite Limits.

13. We shall now consider the case in which the range is infinite. If there be but a finite number of infinities across which $\int f(x)\,dx$ is not convergent, no new point arises. It may indeed happen that

$$\lim_{H=\infty}\int_{-H}^{H} f(x)\,dx$$

is finite and determinate, although $\int_{-\infty}^{\infty}$ is not. If so, we call the former a principal value. We shall not be concerned with this, however; and indeed principal values of this kind are not particularly interesting, and may always be reduced by simple substitutions to those of the kind which has already been considered.

We suppose then that $f(x)$ has an infinity of such infinities,

$$X_1 < X_2 < X_3 \dots \quad (\lim X_i = +\infty);$$

further, that each of them is an infinity X^1 (§ 8), and that there is a positive constant H, such that

$$X_{i+1} - X_i > H \quad (i = 1, 2, \dots).$$

Then

$$P\int_a^x f(x)\,dx \tag{1}$$

is determinate for any finite value of $x > a$ and distinct from any X_i; in particular, for values of $x > a$ which satisfy the conditions

$$|x - X_i| > \delta \quad (i = 1, 2, \dots), \tag{c}$$

where δ is any small fixed positive quantity.

Then, *if, when x tends to ∞ through any system of values which satisfy condition* (c), *the principal value* (1) *tend, however small be δ, to a finite limit independent of the particular system chosen, this limit* (which must evidently be independent of δ) *will be called* the principal value of the integral $\int_a^\infty$, *and denoted by*

$$P\int_a^\infty f(x)\,dx.$$

Similarly for an infinite lower limit.

14. We shall shortly give a still more general definition, which may be used when condition (c) cannot be satisfied, or when $f(x)$ has infinitely many infinities X^1 within a finite range. But we may first illustrate the preceding definition by some examples.

15. (i.) Let

$$f(x) = \frac{\sin ax}{\sin x}\,\frac{1}{\theta^2 + x^2}.$$

Here $X_i = i\pi$. Suppose $0 < \delta < \frac{1}{2}\pi$. Then

$$P\int_0^{N\pi+\delta} = \int_0^{\frac{1}{2}\pi} + \sum_1^N P\int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi} - \int_{N\pi+\delta}^{(N+\frac{1}{2})\pi}. \tag{1}$$

Also

$$P\int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi} = P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (-)^i \frac{\sin a(x+i\pi)}{\sin x}\,\frac{dx}{\theta^2 + (x+i\pi)^2}$$

$$= (-)^i \cos ai\pi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sin ax}{\sin x}\,\frac{dx}{\theta^2 + (x+i\pi)^2}$$

$$+ (-)^i \sin ai\pi\, P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos ax}{\sin x}\,\frac{dx}{\theta^2 + (x+i\pi)^2}.$$

We can determine a finite constant L, such that

$$\left|\frac{\sin ax}{\sin x}\right| < L, \quad -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi;$$

and then the modulus of the first term is

$$< \frac{L}{\theta^2 + \{(i-\frac{1}{2})\pi\}^2},$$

the general term of a convergent series. Again, the conditions of the lemma of § 12 are satisfied by the principal value in the second term, if we put

$$f(x) = \frac{1}{\sin x}, \quad \phi(x) = \frac{\cos ax}{\theta^2 + (x+i\pi)^2}.$$

Also
$$P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{dx}{\sin x} = 0.$$

Hence

$$P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos ax}{\sin x}\,\frac{dx}{\theta^2+(x+i\pi)^2} = \phi'(\mu)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{x}{\sin x}\,dx \quad (-\tfrac{1}{2}\pi \leq \mu \leq \tfrac{1}{2}\pi)$$

$$= \left[-\frac{a\sin a\mu}{\theta^2+(\mu+i\pi)^2} - \frac{2\cos a\mu\,(\mu+i\pi)}{\{\theta^2+(\mu+i\pi)^2\}^2}\right]K,$$

i.e.,
$$< \left[\frac{a}{\theta^2+\{(i-\frac{1}{2})\pi\}^2} + \frac{2(i+\frac{1}{2})\pi}{\{\theta^2+[(i-\frac{1}{2})\pi]^2\}^2}\right]K,$$

in absolute value, K being some finite constant. And this again is the general term of a convergent series.

Hence $P\int_0^{N\pi+\delta}$ tends to a finite limit for $N=\infty$. For the last term of (1) evidently tends to zero. And so does $\int_{N\pi+\delta}^{x}$, $[N\pi+\delta < x < (N+1)\pi-\delta]$. Hence, according to our definition,

$$P\int_0^{\infty} \frac{\sin ax}{\sin x}\,\frac{dx}{\theta^2+x^2}$$

is convergent. As a matter of fact its value, if $a<1$, $\theta>0$, is

$$\frac{\pi}{2\theta}\,\frac{\sinh a\theta}{\sinh\theta}.$$

16. (ii.) We shall now establish the convergence of a general class of principal values which includes the preceding example as a particular case. In the general case, however, we are obliged to use an argument which is not quite so simple.

A General Convergence Theorem.

THEOREM.—*If $\psi(x)$ be a function which possesses a continuous derivate $\psi'(x)$ for all positive values of x, and $\psi(x)$, $\psi'(x)$ tend* steadily *to zero*

for $x=\infty$, *then*

$$P\int^{\infty}\frac{\sin ax}{\sin x}\psi(x)\,dx,\quad P\int_0^{\infty}\frac{\cos ax}{\cos x}\psi(x)\,dx$$

*will be convergent, provided a be not an integer.**

Take, for instance, the former of the two. It will be clear, after the discussion of the preceding section, that it is sufficient for our purpose to prove that we can choose N so large that

$$\left|\sum_{N+1}^{N'}P\int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi}\right|<\sigma,$$

for all values of $N'>N$; σ being an arbitrarily small positive quantity. Now

$$\begin{aligned}\sum_{N+1}^{N'}P\int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi} &= \sum_{N+1}^{N'}(-)^i P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\sin a(x+i\pi)}{\sin x}\psi(x+i\pi)\,dx\\ &= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\sin ax}{\sin x}\sum_{N+1}^{N'}(-)^i\cos ai\pi\,\psi(x+i\pi)\,dx\\ &\quad+P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\cos ax}{\sin x}\sum_{N+1}^{N'}(-)^i\sin ai\pi\,\psi(x+i\pi)\,dx. \quad (1)\end{aligned}$$

Now the series

$$\sum^{\infty}(-)^i\cos ai\pi\,\psi(x+i\pi),\quad \sum^{\infty}(-)^i\sin ai\pi\,\psi(x+i\pi)$$

are *uniformly convergent* for values of x in $(-\frac{1}{2}\pi,\frac{1}{2}\pi)$, if a is not an odd integer.

For, by a well known lemma due to Abel,

$$\sum_{n}^{n'}u_iv_i=\sum_{n}^{n'-1}(u_i-u_{i+1})V_i+u_{n'}V_{n'},$$

if $$V_i=v_n+v_{n+1}\ldots+v_i.$$

Let, for instance, $v_i=(-)^i\cos ai\pi$, $u_i=\psi(x+i\pi)$.

Then V_i oscillates between finite limits as i increases. And, owing to the conditions imposed upon $\psi(x)$,

$$\sum(u_i-u_{i+1})$$

---

* If a is an odd integer, the integrals are of the ordinary kind, and are convergent or divergent according as

$$\int_0^{\infty}\psi(x)\,dx$$

is convergent or divergent. If a is an even integer, the first of them is an ordinary integral, and convergent; the second is a principal value, and convergent.

converges *absolutely* and *uniformly*, and u_i tends uniformly to zero. Hence $\Sigma u_i v_i$ converges uniformly. A similar proof applies if

$$v_i = (-)^i \sin ai\pi.$$

It follows at once that we can make the modulus of the first term of (1) assignedly small by choice of N. And the second, by the lemma of § 12, used as in the preceding section, is

$$\left[\cos ax \sum_{N+1}^{N'} (-)^i \sin ai\pi\, \psi(x+i\pi)\right]'_{x=\mu} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{x}{\sin x}\, dx,$$

where $$-\tfrac{1}{2}\pi \leqq \mu \leqq \tfrac{1}{2}\pi.$$

The quantity in square brackets is

$$-a \sin a\mu \sum_{N+1}^{N'} (-)^i \sin ai\pi\, \psi(\mu+i\pi) + \cos a\mu \sum_{N+1}^{N'} (-)^i \sin ai\pi\, \psi'(\mu+i\pi).$$

Now $$\sum^{\infty} (-)^i \sin ai\pi\, \psi'(a+i\pi)$$

is also uniformly convergent, by the same argument as before. Hence the modulus of the second term of (1) can also be made assignedly small by choice of N. And so the theorem follows.

A similar conclusion holds for

$$P\int_0^{\infty} \frac{\sin ax}{\cos x}\, \psi(x)\, dx, \quad P\int_c^{\infty} \frac{\cos ax}{\sin x}\, \psi(x)\, dx \quad (0 < c),$$

$$P\int_0^{\infty} \sin ax \tan x\, \psi(x)\, dx, \;\ldots,$$

$$P\int_0^{\infty} \frac{\sin ax}{\sin x - \sin a}\, \psi(x)\, dx,$$

$$P\int_0 \frac{\cos ax}{\cos x - \cos a}\, \psi(x)\, dx \quad (0 < a < \pi),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots.$$

Thus $\psi(x)$ may be, for example,

$$x^{-\mu} \quad (0 < \mu < 1),$$

$$e^{-\lambda x}, \quad \frac{1}{x+\theta}, \quad \frac{x}{x^2+\theta^2}, \quad x^{\mu} e^{-\lambda x} \quad (\mu > -1), \quad \ldots.$$

It is obviously sufficient if the conditions as to the *steady decrease* of $\psi(x)$, $\psi'(x)$ are satisfied after some finite value of x.

Principal values of these types are interesting in many ways. Cauchy evaluated some of them, as *e.g.*,

$$P\int_0^\infty \frac{\sin ax}{\sin bx}\,\frac{dx}{1+x^2},\quad P\int_0^\infty \frac{\cos ax}{\cos bx}\,\frac{dx}{1+x^2};$$

but he never defined precisely the sense in which they are convergent; nor, so far as I am aware, has any later writer done so.

A more general Theorem.

17. (iii.) THEOREM.—*If $\psi(x)$ satisfy the conditions of the preceding section, and $\phi(u)$ be a function which has a continuous derivate for all values of u, $0 \leqq u \leqq 1$, then*

$$P\int^\infty \frac{\cos ax}{\cos x}\,\phi(\cos^2 x)\,\psi(x)\,dx,\quad P\int^\infty \frac{\sin ax}{\sin x}\,\phi(\sin^2 x)\,\psi(x)\,dx$$

will be convergent.

Arguing as before, we obtain, instead of equation (1) of § 16,

$$\sum_{N+1}^{N'} P\int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi} = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sin ax}{\sin x}\,\phi(\sin^2 x)\sum_{N+1}^{N'}(-)^i\cos ai\pi\,\psi(x+i\pi)\,dx$$

$$+P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos ax}{\sin x}\,\phi(\sin^2 x)\sum_{N+1}^{N'}(-)^i\sin ai\pi\,\psi(x+i\pi)\,dx;$$

and, as before, the first line can be made assignedly small by choice of N, and the second is (using the lemma of § 12 once more)

$$\left[\cos ax\,\phi(\sin^2 x)\sum_{N+1}^{N'}(-)^i\sin ai\pi\,\psi(x+i\pi)\right]'_{x=\mu}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{x}{\sin x}\,dx.$$

The conclusion follows as before.

It would not be difficult to generalize these theorems further; but what we have proved will be sufficient for our present purpose. We may mention, among formulæ of other types, the two

$$\left.\begin{aligned} P\int_0^\infty \frac{dx}{\cos x - px\sin x} &= 0 \\ P\int_0^\infty \frac{1}{\cos x - px\sin x}\,\frac{dx}{a^2+x^2} &= \frac{\pi}{2a(\cosh a + pa\sinh a)} \end{aligned}\right\}\quad (p>0).$$

These follow easily from Cauchy's theorem. We have only to observe that the roots of

$$\cot x = px \quad (p>0)$$

are all real.

Transformation of Principal Values.

18. We shall now consider the question of the transformation of a principal value by the substitution of a new variable. We shall begin by considering the case of a principal value defined as in § 4. When we attempt to apply the same process to the case in which the limits are infinite, and the principal value defined as in § 13, we shall find that the definitions already given are inadequate. We shall thus be led to a more general definition.

19. Let us suppose that $P\int_a^A f(x)\,dx$ is convergent, the range of integration including one, and only one, infinity X^1, viz., $x = X$; and that, if ξ be any positive quantity, however small,

$$\int_a^{X-\xi}, \quad \int_{X+\xi}^{A},$$

can be transformed in the ordinary way by the substitution

$$x = \phi(y), \quad y = \phi^{-1}(x) = \psi(x);$$

finally, that ϕ and its first *two* derivates are continuous in the immediate neighbourhood of $x = X$, and that $\psi'(y)$ is not zero when $x = X$. *Then* $P\int_a^A$ *can be transformed by the ordinary rule*, that is to say,

$$P\int_a^A f(x)\,dx = P\int_b^B f\left[\phi(y)\right]\phi'(y)\,dy.$$

In the proof of this theorem we shall need the following lemma.

20. Lemma.—*If* $\quad f(u) = \Omega_\nu(u)\,\Theta(u),$

where $\Omega_\nu(u)$, $\Theta(u)$ *are functions of the type considered in* §§ 8, 9, 10, *and* κ, κ' *tend to zero in such a way that*

$$\lim \frac{\kappa'-\kappa}{\kappa^{1+\mu}} = 0 \quad (\mu > 0);$$

then $$\lim P\int_{-\kappa}^{\kappa'} f(u)\,du = \lim P\int_{-\kappa}^{\kappa} = 0;$$

and so, if ξ *be small enough,*

$$\lim\left(\int_{-\xi}^{-\kappa} + \int_{\kappa'}^{\xi}\right) = P\int_{-\xi}^{\xi}.$$

For $$\int_{\kappa}^{\kappa'} f(u)\,du = \int_{\kappa}^{\kappa'} \Omega_\nu(u)\,\Theta(u)\,du.$$

Now we may suppose κ, κ' so small that $\Omega_\nu(u)$ does not change its sign in (κ, κ'), and increases as u decreases, and $\kappa' > \kappa$. Then this is

$$\Theta(\lambda)\int_{\kappa}^{\kappa'} \Omega_\nu(u)\,du \quad (\kappa < \lambda < \kappa')$$

$$< \Theta(\lambda)(\kappa' - \kappa)\,\Omega_\nu(\kappa),$$

which tends to zero with κ, κ'.

21. Let $$b = \psi(a), \quad B = \psi(A),$$

and suppose, *e.g.*, $b < B$. Then

$$P\int_a^A f\,dx = \lim_{\xi=0}\left(\int_a^{X-\xi} + \int_{X+\xi}^{A}\right) f(dx)$$

$$= \lim_{\xi=0}\left(\int_b^{\psi(X-\xi)} + \int_{\psi(X+\xi)}^{B}\right) f[\phi(y)]\,\phi'(y)\,dy. \tag{1}$$

Since $\phi'(y)$ is not zero for $x = X$,

$$b < \psi(X-\xi) < \psi(X+\xi) < B.$$

Now $$f[\phi(y)] = \Omega_\nu\{\phi(y) - X\}\,\Theta[\phi(y)].$$

and $\phi(y) - X = \phi(y) - \phi(Y)$, say,

$$= (y-Y)\,\phi'(Y) + \tfrac{1}{2}(y-Y)^2\,\phi''\{Y + \theta(y-Y)\},$$

where $$0 \leqq \theta \leqq 1.$$

Hence $$\frac{1}{x-X} = \frac{1}{(y-Y)\,\phi'(Y)}$$

+ terms which remain continuous at $x = X$.

Also $$l(x-X) = l(y-Y) + l\phi'(Y) + l\left[1 + \tfrac{1}{2}\frac{y-Y}{\phi'(Y)}\,\phi''\right]$$

$$= l(y-Y) + l\phi'(Y) + (y-Y)\,\Theta_1(y),$$

where Θ_1 is continuous.

We shall suppose for simplicity that

$$\Omega_\nu u \equiv \frac{1}{u}\,|\,lu\,|^{\nu_1}\, l_2^{\nu_2} u\,;$$

when there are more factors the argument is more complicated, but in principle the same. Also we shall suppose that c is the greatest integer in r_1,

$$r_1 = s + c \quad (0 \leqq s < 1).$$

Then
$$|\, l\,(x-X)\,|^{r_1}$$
$$= |\, l\,(y-Y)\,|^{r_1} \left| 1 + \frac{l\phi' Y}{l\,(y-Y)} + \frac{(y-Y)\,\Theta_1}{l\,(y-Y)} \right|^{r_1}$$
$$= |\, l\,(y-Y)\,|^{r_1} + \sum_{i=1}^{c+1} a_i \,|\, l\,(y-Y)\,|^{r_1-i}$$
$$+ \,|\, l\,(y-Y)\,|^{s-2}\, p\,(y) + (y-Y)\,|\, l\,(y-Y)\,|^{r_1-1}\, q\,(y),$$

where a_i is a constant, and $p\,(y)$, $q\,(y)$ are continuous functions.

Again,
$$l^s\,(x-X) = l^s\,(y-Y) + l \left\{ 1 + \frac{l\phi'\,(Y)}{|\, l\,(y-Y)\,|} + \frac{(y-Y)\,\Theta_1}{|\, l\,(y-Y)\,|} \right\}$$
$$= l^s\,(y-Y) + \sum_{\kappa=1}^{c+1} \gamma_\kappa \,|\, l\,(y-Y)\,|^{-\kappa} + |\, l\,(y-Y)\,|^{-c-2}\, \lambda\,(y)$$
$$+ (y-Y)\,\mu\,(y),$$

where γ_κ is a constant, and $\lambda\,(y)$, $\mu\,(y)$ are continuous functions.

On forming the product Ω_ν we see that it is the sum of—

(i.) A finite number of terms each of which possesses a convergent integral across $y = Y$. It is to be remembered that

$$(y-Y)^{-1}\,|\, l\,(y-Y)\,|^{\rho}\, l^s\,(y-Y)\, \phi\,(y)$$

is a term of this kind, if $\phi\,(y)$ is continuous and $\rho < -1$; so that these terms include, *e.g.*,

$$(y-Y)^{-1}\,|\, l\,(y-Y)\,|^{s-2}\, \lambda\,(y)$$

and
$$(y-Y)^{-1}\,|\, l\,(y-Y)\,|^{s-2}\, l^s\,(y-Y)\, p\,(y).$$

(ii.) A finite number of terms of the form

$$A\Omega_{\nu'}\,(y-Y),$$

where A is a constant. Hence

$$P\int_b^B f\left[\phi(y)\right]\phi'(y)\,dy$$

is convergent. Also, if

$$\psi(X-\xi) = Y-\eta, \quad \psi(X+\xi) = Y+\eta',$$
$$X-\xi = \phi(Y)-\eta\phi'(Y)+\tfrac{1}{2}\eta^2\phi''(Y-\theta\eta),$$
$$X+\xi = \phi(Y)+\eta'\phi'(Y)+\tfrac{1}{2}\eta'^2\phi''(Y+\theta'\eta'),$$
$$(0 \leqq \theta, \ \theta' \leqq 1).$$

Hence
$$\frac{\eta'-\eta}{\eta^2}$$

remains finite as η, η' tend to zero with ξ; and therefore, by equation (1) and the lemma of § 20,

$$P\int_a^A f(x)\,dx = P\int_b^B f\left[\phi(y)\right]\phi'(y)\,dy.$$

22. Thus, for instance, if $x = y^2$, and $H \neq n\pi$,

$$P\int_0^H \frac{\sin ax}{\sin x}\,\frac{dx}{x^\mu} = 2P\int_0^{\sqrt{H}} \frac{\sin ay^2}{\sin y^2}\,y^{1-2\mu}\,dy \quad (0<\mu<1).$$

If H tend to ∞ through a series of values included in the intervals

$$\{n\pi+\delta, \ (n+1)\pi-\delta\} \quad (n = 1, 2, \ldots),$$

each side of the equation tends to a finite limit, and it is natural to write

$$P\int_0^\infty \frac{\sin ax}{\sin x}\,\frac{dx}{x_\mu} = 2P'\int^\infty \frac{\sin ay^2}{\sin y^2}\,y^{1-2\mu}\,dy. \tag{1}$$

But the right hand cannot be defined as in § 13, since the intervals

$$\{\sqrt{n\pi}, \ \sqrt{(n+1)\pi}\} \quad (n = 1, 2, \ldots)$$

diminish indefinitely as n increases. This suggests that our former definition may be extended.

23. The following general definition includes as particular cases those which we have been considering, and justifies equation (1) of § 22.

A General Definition.

Let $f(x)$ be a function which possesses a convergent integral over any part of an interval (a, A), where A may be ∞, which does not include any one of an infinite series of isolated points X_i,

$$(a < \ldots < X_i < X_{i+1} \ldots, \quad \lim_{i=\infty} X_i = A);$$

while

$$P\int f(x)\,dx$$

is convergent across any point X_i. Let the points X_i be included in a series of open* intervals

$$\xi_{i,\delta},$$

no two of which have any point in common. And suppose that the interval $\xi_{i,\delta}$ depends on a parameter δ, and that as δ tends to zero each of its extremities tends steadily to X_i. Let the remainder of (a, A) be denoted by R_δ.

If $x < A$ be any point of R_δ,

$$P\int_a^x f(x)\,dx \tag{1}$$

is convergent.

Then, *if, when x tends to A through any series of values lying entirely in R_δ,* (1) *tends, however small be δ, to a finite limit independent of the particular series chosen, this limit*—which must evidently be independent of δ—*will be called the principal value of the integral* $\int_a^A$, and will be denoted by

$$P\int_a^A f(x)\,dx.$$

Further generalizations are at once suggested. It is clear, for instance, that we may extend our definition to meet the case in which the infinities X_i form any enumerable set, and that similar definitions are possible for "conditionally convergent" integrals other than principal values. But, for the reasons stated in § 7, I shall not enter into this.

24. I shall conclude this paper with an illustration of the theorem of §§ 19–21, and the definition of § 23. I shall determine

* An *open* interval is an interval which does not include its extremities.

whether
$$P\int_0^\infty \frac{1}{\sin\left(ax+\dfrac{b}{x}\right)}\frac{dx}{x},$$
where
$$a,\ b>0,\quad ab<\tfrac{1}{4}\pi^2,$$
is determinate or not.

The infinities of the subject of integration are
$$x=0,\quad ax=\tfrac{1}{2}n\pi\pm\sqrt{\{(\tfrac{1}{2}n\pi)^2-ab\}}\quad (n=1,2,\ldots);$$
and, except for 0, are infinities X^1. As n increases
$$\tfrac{1}{2}n\pi+\sqrt{\{(\tfrac{1}{2}n\pi)^2-ab\}}$$
tends to ∞,
$$\tfrac{1}{2}n\pi-\sqrt{\{(\tfrac{1}{2}n\pi)^2-ab\}}$$
to zero.

Consider the transformation
$$y=ax+\frac{b}{x},\quad x=\frac{1}{2a}\{y\pm\sqrt{(y^2-4ab)}\}.$$

As η increases from $2\sqrt{(ab)}$ to ∞ the upper value of x increases steadily from $\sqrt{\left(\dfrac{b}{a}\right)}$ to ∞, and the lower value decreases steadily from $\sqrt{\left(\dfrac{b}{a}\right)}$ to 0. Also
$$x'=\frac{1}{2a}\left\{1\pm\frac{y}{\sqrt{(y^2-4ab)}}\right\},$$
$$x''=\mp\frac{2b}{(y^2-4ab)^{\frac{3}{2}}};$$
and these are continuous for all values of $y>2\sqrt{(ab)}$. Finally,
$$\frac{1}{x}\frac{dx}{dy}=\frac{1}{\sqrt{(y^2-4ab)}},\quad y>2\sqrt{(ab)}.$$

Hence, so long as A is distinct from any of the infinities,
$$P\int_{\sqrt{(b/a)}+\epsilon}^{A}\frac{1}{\sin\left(ax+\dfrac{b}{x}\right)}\frac{dx}{x}=P\int_{2\sqrt{(ab)}+\eta}^{aA+b/A}\frac{1}{\sin y}\frac{dy}{\sqrt{(y^2-4ab)}},$$
however large be A, and however small be $\epsilon>0$. Here η is >0, and *tends to zero* with ϵ.

Let δ be an arbitrarily small positive quantity. We can make A tend to ∞ in such a way that $aA+\frac{b}{A}$ tends to ∞ through a series of values entirely included in the intervals

$$\{n\pi+\delta,\ (n+1)\,\pi-\delta\}\quad (n=1,\,2,\,\ldots),$$

and then the limit of the right hand is

$$P\int_{2\surd(ab)+\eta}^{\infty}\frac{1}{\sin y}\,\frac{dy}{\surd(y^2-4ab)};$$

and it is easy to see that the left becomes $P\int_{\surd(b/a)+\epsilon}^{\infty}$ according to the definition of § 18.

The limit of the right hand for $\epsilon=0$ is determinate, and so

$$P\int_{\surd(b/a)}^{A}\frac{1}{\sin\left(ax+\dfrac{b}{x}\right)}\,\frac{dx}{x}=P\int_{2\surd(ab)}^{\infty}\frac{1}{\sin y}\,\frac{dy}{\surd(y^2-4ab)}.$$

Similarly

$$P\int_{0}^{\surd(b/a)}\frac{1}{\sin\left(ax+\dfrac{b}{x}\right)}\,\frac{dx}{x}=P\int_{2\surd(ab)}^{\infty}\frac{1}{\sin y}\,\frac{dy}{\surd(y^2-4ab)}.$$

Finally,

$$P\int_{0}^{\infty}\frac{1}{\sin\left(ax+\dfrac{b}{x}\right)}\,\frac{dx}{x}=2P\int_{2\surd(ab)}^{\infty}\frac{1}{\sin y}\,\frac{dy}{\surd(y^2-4ab)}.$$

This is easily verified by the help of Cauchy's theorem. In fact each integral $=\dfrac{\pi}{2\surd(ab)}$. And more generally, if

$$(n+1)\,\pi>u=2\surd(ab)>n\pi,$$

each of the integrals $=\pi\sum\limits_{-n}^{n}\dfrac{1}{\surd\{u^2-(i\pi)^2\}}$.

Similarly, if $a,\,b>0$.

$$P\int_{0}^{\infty}\frac{1}{\sin\left(ax-\dfrac{b}{x}\right)}\,\frac{dx}{x}=P\int_{-}^{\infty}\frac{1}{\sin y}\,\frac{dy}{\surd(y^2+4ab)}=0.$$

We may perhaps mention the following formulæ of the same kind:—

$$P\int_0^\infty \frac{\sin\left(ax+\frac{b}{x}\right)}{\sin\left(cx+\frac{d}{x}\right)}\frac{dx}{\theta^2+x^2} \quad (\theta>0,\ |c|>|a|,\ |d|>|b|)$$

$$=\frac{\pi}{2\theta}\frac{\sinh\left(a\theta-\frac{b}{\theta}\right)}{\sinh\left(c\theta-\frac{d}{\theta}\right)} \quad (cd<0)$$

$$=\frac{\pi}{2\theta}\frac{\sinh\left(a\theta-\frac{b}{\theta}\right)}{\sinh\left(c\theta-\frac{d}{\theta}\right)}+\frac{\pi\sinh\left(a\sqrt{\frac{d}{c}}-b\sqrt{\frac{c}{d}}\right)}{d-c\theta^2}$$

$$\left(0<cd<\frac{\pi^2}{4}\right). \quad (1)$$

$$P\int_0^\infty \frac{\sin\left(ax+\frac{b}{x}\right)}{\sin\left(cx+\frac{d}{x}\right)}\frac{dx}{x^2-\theta^2} \quad \left[\theta>0,\ |c|>|a|,\ |d|>|b|,\ \sin\left(c\theta+\frac{d}{\theta}\right)\neq 0\right]$$

$$=0 \quad (cd<0)$$

$$=\frac{\pi\sinh\left(a\sqrt{\frac{d}{c}}-b\sqrt{\frac{c}{d}}\right)}{d+c\theta^2} \quad \left(0<cd<\frac{\pi^2}{4}\right). \quad (2)$$

These and many other similar formulæ, which may all be proved by means of Cauchy's theorem, afford more examples of the use of the definitions of this paper.

On the Composition of Group-Characteristics. By W. BURNSIDE.
Received and communicated March 14th, 1901.

In my paper "On Group-Characteristics" (*Proc. Lond. Math. Soc.*, Vol. XXXIII., p. 146) I have already given a short account of what Herr Frobenius had called their "composition."*

In the present communication I consider in greater detail the system of relations of the form

$$G_i G_j = \Sigma g_{ijk} G_k,$$

which indicate how the various irreducible representations of a group combine among themselves. The main result arrived at, which is, I believe, new, indicates how from the complete system of relations of the above form the existence of each self-conjugate sub-group which the group possesses may be deduced.

If G is a group with r sets of conjugate operations, the r distinct representations of the group as an irreducible group of linear substitutions will be denoted by $G_1, G_2, \ldots, G_r$. Of these G_1 will always be used to denote that representation in which each operation corresponds to identity. In G_i the characteristics of the conjugate sets are

$$\chi_1^i, \chi_2^i, \ldots, \chi_r^i,$$

and the number of variables is χ_1^i.

I recall briefly the process on which the so-called composition depends.

Let G_i and G_j be actually set up as groups of linear substitutions in the two distinct sets of variables

$$x_1, x_2, \ldots, x_{\chi_1^i};$$

and

$$y_1, y_2, \ldots, y_{\chi_1^j}.$$

To every operation of G there will then correspond a definite linear substitution on the $\chi_1^i \chi_1^j$ products of the x's and y's, so that G is thus

* The process made use of by Herr Frobenius to obtain the composition of characteristics is given in a slightly different connection by M. Jordan (*Traité des Substitutions*, p. 221).

represented as a group of linear substitutions in $\chi_1^i\chi_1^j$ variables. This representation will not, in general, be irreducible. Suppose it resolved into its irreducible components, and denote by g_{ijk} the number of times that G_k occurs (for each k from 1 to r). Each symbol g_{ijk} is either zero or a positive integer. The sum of the multipliers of any operation of the p-th conjugate set in this representation will then be $\sum_{k=1}^{k=r} g_{ijk}\chi_p^k$. On the other hand, the sum of these multipliers is $\chi_p^i\chi_p^j$, in consequence of the manner in which the representation has been constructed from G_i and G_j.

Hence for each p we have the equation

$$\chi_p^i\chi_p^j = \sum_1^r g_{ijk}\chi_p^k.$$

This system of r equations among the χ's will now be represented by the single symbolical equation

$$G_i G_j = \sum_1^r g_{ijk} G_k = G_j G_i. \qquad \text{(i)}$$

The complete system of r^2 equations of this form, for $i, j = 1, 2, \ldots, r$, may be primarily regarded as giving in a succinct form the result of combining any two of the irreducible representations of G by the process used above. From this point of view $G_i G_j$ may be regarded as a symbol for the group of linear substitutions on $\chi_1^i\chi_1^j$ variables constructed as above; and the notation may be extended to give a definite meaning to any symbol of the form $G_i^a G_j^b \ldots G_k^c$, or to the sum of any number of such symbols.

Equations (i) may, however, be looked at from another point of view; viz., as giving the multiplication table of a set of complex commutative numbers. In fact, if G_i, G_j, G_k be set up on the sets of variables

$$x_1, x_2, \ldots, x_{\chi_1^i},$$

$$y_1, y_2, \ldots, y_{\chi_1^j},$$

$$z_1, z_2, \ldots, z_{\chi_1^k},$$

and the resulting group of linear substitutions on the $\chi_1^i\chi_1^j\chi_1^k$ products of the x's, y's and z's be resolved into its irreducible components, the number of times that G_l occurs may be represented either by $\sum_p g_{ijp}g_{pkl}$ or by $\sum_p g_{ikp}g_{pjl}$ or by $\sum_p g_{jkp}g_{pil}$. These three numbers must therefore be the same whatever i, j, k, and l may be. But these con-

ditions are sufficient to ensure that when $G_i G_j \,.\, G_k$ and $G_i \,.\, G_j G_k$ are calculated from equations (i) they shall have the same value.

Equations (i) are therefore a consistent system for defining the multiplication of a set of r complex commutative numbers as stated.

The analogy in form between equations (i) and the equations*

$$C_i C_j = \sum_1^r c_{ijk} C_k = C_j C_i, \tag{ii}$$

which express the way in which the conjugate sets of G combine among themselves, is complete. Moreover the latter set may be regarded as giving the multiplication table of a set of commutative complex numbers.†

Now with the system of equations (ii) what is ordinarily called the composition of the group is intimately connected. In fact, the necessary and sufficient condition that G may have a self-conjugate sub-group is that it may be possible to select a set of the C's (less than the whole) which combine by multiplication among themselves. The totality of the operations belonging to such a set then constitute a self-conjugate sub-group.

It is natural to ask whether the system of equations among the G's have not a similar connexion with the composition of the group. Let Γ be a self-conjugate sub-group of G. Among the irreducible representations of G there must be a certain number in which every operation of Γ corresponds to the identical operation (*Proc. Lond. Math. Soc.*, Vol. XXIX., pp. 563, 564). If G_i and G_j be two of these, then in the group formed from G_i and G_j by the above process of composition, every operation of Γ corresponds to the identical operations; and this is therefore true for every irreducible component that occurs in $\sum_k g_{ijk} G_k$ with a non-zero coefficient.

Hence the totality of the G's, in which the operations of Γ correspond to the identical operation, combine among themselves by multiplication. The converse of this result will now be shown to be true. The result to be proved may be stated as follows :—

THEOREM.—If a number of the irreducible representations of G (less than the whole) combine among themselves by multiplication, then G has a self-conjugate sub-group, whose operations correspond, in these representations and in these only, to the identical operation.

---

* "Group-Characteristics" (p. 148).
† *Loc. cit.* (p. 148).

The proof of this theorem is facilitated by the following lemma:—

Let $X_1, X_2, \ldots, X_m$ be m linear functions of the n variables

$$x_1, x_2, \ldots, x_n;$$

subject to the sole condition that no one X is a multiple of any other. Then, if these functions be formed with s distinct sets of variables

$$\begin{array}{l} x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}; \\ x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}; \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_1^{(s)}, x_2^{(s)}, \ldots, x_n^{(s)}; \end{array}$$

it is always possible to ensure, by taking s sufficiently large, that the m products

$$X_1^{(1)} X_1^{(2)} \ldots X_1^{(s)}, \quad X_2^{(1)} X_2^{(2)} \ldots X_2^{(s)}, \quad \ldots, \quad X_m^{(1)} X_m^{(2)} \ldots X_m^{(s)}$$

are linearly independent.

Suppose that, when $s = t-1$, just k of the products are linearly independent; so that for each suffix i

$$X_i^{(1)} X_i^{(2)} \ldots X_i^{(t-1)} = \sum_{j=1}^{j=k} a_{ij} P_j,$$

where $P_1, P_2, \ldots, P_k$ are linearly independent. When $s = t$, the m products can be expressed linearly in terms of

$$P_a X_b^{(t)} \quad (a = 1, 2, \ldots, k;\ b = 1, 2, \ldots, m).$$

If only k of these products were linearly independent, the various $X^{(t)}$'s which multiply any one P must be multiples of each other. Also, since by supposition $k < m$, there must be at least one P which is multiplied by two different $X^{(t)}$'s. Hence the supposition that the number of linearly independent t-products is equal to the number of independent $(t-1)$-products involves that some one $X^{(t)}$ is a multiple of some other, contrary to the supposition made. The number of linearly independent t-products is therefore greater than the number of independent $(t-1)$-products; and hence by increasing t sufficiently a finite integer s may be found such that the m s-products are linearly independent.

Suppose now that G' is an irreducible group of linear substitutions in m variables, of order n'. Unless G' contains operations which multiply each variable by the same root of unity (*i.e.*, self-conjugate

operations), it is always possible to choose a linear function

$$a_1 x_1 + a_2 x_2 + \dots + a_m x_m$$

of the variables which is not changed into a multiple of itself by any operation of G'. In fact, in order that this linear function may be changed into a multiple of itself by some operation S, one or more relations must hold among the a's; and it is therefore only necessary to choose the a's so that no one of a finite number of linear equations connecting them is satisfied.

Hence, if G' has no self-conjugate operations, n' linear functions of the variables

$$X_1,\ X_2,\ \dots,\ X_{n'}$$

may be found which are regularly permuted among themselves when the variables undergo the n' operations of the group, while also no one X is a multiple of another. These n' functions will not be linearly independent; but, by the lemma, it is possible to choose s so that the s-products

$$X_1^{(1)} X_1^{(2)} \dots X_1^{(s)},\quad X_2^{(1)} X_2^{(2)} \dots X_2^{(s)},\quad \dots,\quad X_{n'}^{(1)} X_{n'}^{(2)} \dots X_{n'}^{(s)}$$

formed from s distinct sets of variables are linearly independent, while they are regularly permuted among themselves when each set of variables undergoes simultaneously the substitutions of G'. Hence, among the component groups that occur in G^s, the representation of G' as a group of regular permutations is one. But this when reduced to its irreducible components contains every irreducible representation of G'. Among the groups which arise from G' by successive multiplications by itself, every possible irreducible representation of G' will therefore occur.

Next suppose that G' has self-conjugate operations. Since they multiply each variable by the same root of unity, they must constitute a cyclical sub-group. Let p be the order of this sub-group, ω a primitive p-th root of unity, and S a self-conjugate operation which generates the sub-group. Then in this case $n'' \ (= n'/p)$ linear functions of the variables

$$X_1,\ X_2,\ \dots,\ X_{n''}$$

may be found, no one of which is a multiple of another, with the following properties. Each is changed into ω times itself by S, and every operation of G' which is not self-conjugate permutes the p-th powers of these functions regularly. Suppose now s chosen so large that

$$X_1^{(1)} X_1^{(2)} \dots X_1^{(s)},\quad X_2^{(1)} X_2^{(2)} \dots X_2^{(s)},\quad \dots,\quad X_{n''}^{(1)} X_{n''}^{(2)} \dots X_{n''}^{(s)}$$

are linearly independent; and further take s to be a multiple of p. Then the n' functions

$$X_i^{(1)} X_i^{(2)} \dots X_i^{(s)} \Big[x_0^{p-1} + \omega x_0^{p-2} X_i^{(s+1)} + \omega^2 x_0^{p-3} X_i^{(s+1)} X_i^{(s+2)} + \dots \\ \dots + \omega^{p-1} X_i^{(s+1)} X_i^{(s+2)} \dots X_i^{(s+p-1)} \Big]$$

$$\left(\begin{array}{l} i = 1, 2, \dots, n''; \\ \omega = \text{each } p\text{-th root of unity, including unity} \end{array} \right)$$

are linearly independent; and they are regularly permuted among themselves when the $s+p-1$ sets of variables simultaneously undergo the operations of G', while x_0 is unaltered, *i.e.*, undergoes the identical operation only. Hence, among the components of

$$G''^{s} G_1^{p-1} + G''^{s+1} G_1^{p-2} + \dots + G''^{s+p-1},$$

the representation of G' as a group of regular permutations occurs. Now G_1 must arise at some stage when G' is repeatedly multiplied by itself. Hence in this case again, among the groups which arise from G' by successive multiplication by itself, every possible irreducible representation of G' must occur.

Suppose now that G_i and G_j are any two of the irreducible representations of G; and that H_i and H_j are the self-conjugate sub-groups of G whose operations are represented by identity in G_i and G_j respectively. Construct, by the process which has just been investigated, a function of s (sufficiently large) sets of variables which takes n_i linearly independent values for the operations of G_i, n_i being the order of G/H_i; and construct a similar function in t sets for G_j. Call these functions P and Q. The product PQ will then remain unaltered only for those operations of G which are common to H_i and H_j. If H_{ij} is the group common to H_i and H_j, PQ will take n_{ij} distinct values for the operations of G, n_{ij} being the order of G/H_{ij}; and, since the P's and Q's are linearly independent, these n_{ij} products are so also. Hence, among the components of $G_i^s G_j^t$, there occurs a group of regular permutations, simply isomorphic with G/H_{ij}; and therefore, by the repeated multiplication of G_i and G_j, every irreducible representation of G will arise in which the operations of H_{ij} correspond to identity, and no others.

From these results the truth of the theorem stated above follows at once. Suppose in fact that, of the irreducible representations $G_1, G_2, \dots, G_r$ of a group G, a number less than r, say

$$G_1,\ G_2,\ \dots,\ G_s,$$

combine by multiplication among themselves. No one of these can

be simply isomorphic with G, since from such a representation every other one arises by multiplication. Moreover, if $H_1, H_2, \ldots, H_s$ are the self-conjugate sub-groups of G which correspond to the identical operation in $G_1, G_2, \ldots, G_s$, then $H_1, H_2, \ldots, H_s$ must have a common sub-group H, as otherwise, by the last result, s would be equal to r. Finally, if H is the greatest common sub-group of $H_1, H_2, \ldots, H_s$, then from $G_1, G_2, \ldots, G_s$ all possible irreducible representations of G/H arise; and therefore the whole of the operations of H cannot correspond to identity in any of the remaining representations $G_{s+1}, G_{s+2}, \ldots, G_r$. The theorem is therefore completely proved.

In illustration of the foregoing the multiplication tables of the G's for the octahedral and icosahedral group have been calculated.* Each of these groups has five sets of conjugate operations, and therefore five distinct irreducible representations. For the octahedral group the numbers of variables in the distinct forms are 1, 1, 2, 3, 3. If the corresponding representations are denoted by G_1, G_2, G_3, G_4, and G_5, then, for each t,

$$G_1 G_t = G_t,$$

and the remaining products are given by the table

	G_2	G_3	G_4	G_5
G_2	G_1	G_3	G_5	G_4
G_3		$G_1+G_2+G_3$	G_4+G_5	G_4+G_5
G_4			$G_1+G_3+G_4+G_5$	$G_2+G_3+G_4+G_5$
G_5				$G_1+G_3+G_4+G_5$

For the icosahedral groups the numbers of the variables are 1, 3, 3, 4, 5, and the corresponding table is

	G_2	G_3	G_4	G_5
G_2	G_1+2G_4	G_4+G_5	$G_3+G_4+G_5$	$G_2+G_3+G_4+G_5$
G_3		G_1+2G_4	$G_2+G_4+G_5$	$G_2+G_3+G_4+G_5$
G_4			$G_1+G_2+G_3+G_4+G_5$	$G_2+G_3+G_4+2G_5$
G_5				$G_1+G_2+G_3+2G_4+2G_5$

The table for the octahedral group shows that G_1 and G_2 combine

* The calculation of these tables involves only the solution of simple equations when the characteristics are known. The determination of the latter present no difficulties in the present cases. They are given (p. 1012) in Herr Frobenius' memoir "Ueber Gruppencharaktere" (*Berliner Sitzungsberichte*, 1896).

among themselves, as also do G_1, G_2, and G_3. Every operation of the tetrahedral group which is contained as a self-conjugate sub-group is represented by identity in G_1 and G_2; and every operation of the quadratic self-conjugate sub-group of order 4 is represented by identity in G_1, G_2, and G_3.

From the table for the icosahedral group it is clear that no set of G's can be chosen which combine among themselves, and this agrees with the fact that the icosahedral group is simple.

In conclusion it may be noticed that a single relation among the G's will sometimes be sufficient to indicate the existence of a self-conjugate sub-group. Thus, if such a relation as

$$G_i G_j = \chi_1^i G_j$$

holds, those G's which arise from the repeated multiplication of G_i by itself must combine among themselves. In fact they all, when multiplying G_j, reproduce G_j, and they therefore cannot constitute the complete system of G's. Rather more generally, if a, β, γ, &c., are positive integers, the totality of G_i's for which such a relation as

$$G_i (aG_a+\beta G_b+\gamma G_c+\ldots) = \chi_1^i (aG_a+\beta G_b+\gamma G_c+\ldots)$$

holds combine among themselves. Thus, for the octahedral group G_4+G_5 is such a combination, which is reproduced on multiplication by G_1, G_2, and G_3.

Thursday, April 11th, 1901.

Dr. HOBSON, F.R.S., President, in the Chair.

Twelve members present.

Mr. Martin Adlard, B.A., Mathematical Master, King's School, Ely, and Mr. James Hopwood Jeans, B.A., Scholar of Trinity College, Cambridge, were elected members.

Mr. Basset made a short communication "On the Projective Properties of Cubic and Quartic Curves." Mr. Love also spoke on the subject.

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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

EDITED BY R. TUCKER AND A. E. H. LOVE.

VOL. XXXIV.—NOS. 767—771.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contributions which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of ten guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the year 1902 are the *second* Thursdays in January, February, March, April, May, June, November, and December.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Communications for the Secretaries may be forwarded to them at the following addresses:—

London Mathematical Society, 22 Albemarle Street, W.
24 Hillmarton Road, West Holloway, N. } R. TUCKER.
34 St. Margaret's Road, Oxford.—A. E. H. LOVE.

Lt.-Col. Cunningham announced the factorisation of the algebraic prime factors of $5^{75}-1$ and $5^{105}-1$. The former

$$= 151 \,.\, 3301 \,.\, 183794551 \,.\, 99244414459501,$$

and the latter

$$= 21226783250214361 \,.\, 207468970805907721.$$

The composition of the three large factors has not been determined.

A paper by Dr. F. Morley, entitled "Summation of the Series $\Sigma_0^\infty \Gamma^3 (a+n)/\Gamma^3 (1+n)$," was communicated from the Chair.

The following presents were made to the Library:—

"Educational Times," April, 1901.

"Indian Engineering," Vol. XXIX., Nos. 8–10, Feb. 23–March 9, 1901.

"Nautical Almanac for 1904," 8vo; Edinburgh, 1901.

"Periodico di Matematica," Serie 2, Vol. III., Fasc. 5; "Supplemento," Anno IV., Fasc. 5; Livorno, 1901.

"Proceedings of the American Philosophical Society," Vol. XXXIX., No. 164, Oct.–Dec., 1900.

"Le Matematiche, pure ed applicate," Vol. I., Num. 1; 1901.

Various papers by Carl Størmer:—

"Quelques théorèmes sur l'équation de Pell

$$x^2 - Dy^2 = \pm 1$$

et leurs applications"; Christiania, 1897.

"Om en generalisation af integralet

$$\int_0^\infty \frac{\sin ax}{x}\, dx = \frac{\pi}{2}\text{''};$$

Christiania, 1895.

"Sur une propriété arithmétique des logarithmes des nombres algébriques," Paris, 1900.

"Sur une équation indéterminée"; Paris, 1898.

"Solution complète en nombres entiers de l'équation

$$m \operatorname{arc\,tan} \frac{1}{x} + n \operatorname{arc\,tan} \frac{1}{y} = k\, \frac{\pi}{4}\text{''};$$

Paris, 1899.

Various papers by N. J. Hatzidakis:—

"Displacements depending on One, Two, ..., k Parameters in a Space of n Dimensions," 4to.

"Sur les équations cinématiques fondamentales des variétés dans l'espace à n dimensions" (*Comptes Rendus*, 1900).

"*Συμβόλη εἰς τὴν διαφοφίκην γεωμέτριαν τῶν n διαστασέων*," Tom I.B, and *βιβλιοκρισιαί*; Athens, 1900.

"Remarque sur une formule de M. Pirondini"; 1899.

"Sur quelques points de la terminologie mathématique."

"Deux démonstrations nouvelles des théorèmes d'Euler et de Meusnier"; 1899.

"Electricité et Optique: la Lumière et ses Théories électrodynamiques," par H. Poincaré (2nd edition, L. Blondin et E. Néculcéa, Paris, 1901).

The following exchanges were received :-

"Proceedings of the Royal Society," Vol. LXVIII., No. 443; 1901.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXV., Hefte 2, 3; Leipzig, 1901.

"Bulletin de la Société Mathématique de France," Tome XXIX., Fasc. 1; Paris, 1901.

"Bulletin of the American Mathematical Society," Vol. VII., No. 6; New York, March, 1901.

"Bulletin des Sciences Mathématiques," Tome XXIV., Dec.; Paris, 1900.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. X., Fasc. 5; Roma, 1901.

"Proceedings of the Physical Society," Vol. XVII., Pt. 5: March, 1901.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Math.-Phys. Klasse, Heft 4; 1900.

Thursday, May 9th, 1901.

Dr. HOBSON, F.R.S., President, in the Chair.

Fifteen members present.

At the President's request Prof. Elliott spoke briefly on the loss the Society had sustained by the death of Mr. C. E. Bickmore (elected February 11th, 1875).

Dr. John Alexander Third was elected a member of the Society, and Prof. Steggall was admitted into the Society.

Major MacMahon communicated two notes: (i.) "On the Series whose Terms are the Cubes and Higher Powers of the Binomial Coefficients," and (ii.) "A Case of Algebraic Partitionment."

Mr. J. B. Dale read a paper on "The Product of Two Spherical Surface Harmonic Functions."

Mr. Macdonald communicated a "Note on the Zeros of the Spherical Harmonic $P_n^{-m}(\mu)$.

A note on "A Property of Recurring Series," by Mr. G. B. Mathews, was read by title.

The following presents were made to the Library :—

"Educational Times," May, 1901.

"Indian Engineering," Vol. XXIX., Nos. 11-15, March 16-April 13, 1901.

"Wiadomósci Matematyczne," Tom V., Zeszyt 1-3; Warsaw, 1901.

Valentin, G. (offprints from "Bibliotheca Mathematica"):—

"Eine Seltene schrift über Winkeldreitheilung."—"Die Beiden Euclid Ausgaben des Jahres 1482"; Berlin, 1893.

"Die Frauen in den exakten Wissenschaften"; Berlin, 1895.

"Beitrag zur Bibliographie der Euler'schen Schriften"; Berlin, 1898.

"Die Vorarbeiten für die allgemeine mathematische Bibliographie"; Leipzig.

Valentin, G.—"De aequatione algebraica quae est inter duas variabiles, in quandam formam canonicam transformata," pamphlet, 4to; Berolini. (Dissertation for Ph.D. degree at Berlin, July 5th, 1879.)

"Mathematical Gazette," Vol. II., No. 27; May, 1901.

"Supplemento al Periodico di Matematica," Anno IV., Fasc. 6, Aprile 1901; Livorno.

"Annals of Mathematics," Series 2, Vol. II., No. 3; Harvard University, 1901.

"Annales de la Faculté des Sciences," Série 2, Tome II.; Toulouse, 1900.

"Washington Observations, 1891-1892," 4to; Washington, 1899.

"A Binary Canon, showing Residues of Powers of 2 for Divisors under 1000, and Indices to Residues," compiled by Lt.-Col. Allan Cunningham, R.E., under the auspices of a British Association Committee; London, 1900. From the author.

The following exchanges were received:—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXV., Heft 4: Leipzig, 1901.

"Bulletin of the American Mathematical Society," Series 2, Vol. VII., No. 7; New York, April, 1901.

"Monatshefte für Mathematik und Physik," Jahrgang XII., Parts 2 and 3; Wien, 1901.

"Bulletin des Sciences Mathématiques," Tome XXV., Jan., Fév.; Paris, 1901.

"Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. VII., Fasc. 3; Napoli, 1901.

"Atti dell'Accademia delle Scienze Fisiche e Matematiche," Vol. X.; Napoli, 1901.

"Journal für die reine und angewandte Mathematik," Bd. CXXIII., Heft 2; Berlin, 1901.

"Archives Néerlandaises," Serie 2, Tome IV., Liv. 2; La Haye, 1901.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. X., Fasc. 6, 7; Roma, 1901.

"Journal of the Institute of Actuaries," Vol. XXXVI., Pt. 1, No. 201; 1901.

"Proceedings of the Cambridge Philosophical Society," Vol. XI., Pt. 2; 1901.

Note on the Zeros of the Spherical Harmonic $P_n^{-m}(\mu)$ *By* H. M. MACDONALD. Received and read May 9th, 1901.

In a former communication* to the Society it was proved that the values of n for which $P_n^{-m}(\cos\theta)$ vanishes, where m is a real positive quantity, decrease as θ increases from 0 to π. The object of the following note is to point out some properties of the function depending on this.

It is known† that when θ is very near to π the values of n for which $P_n^{-m}(\cos\theta)$ vanishes differ from $m+k$, where k is a positive integer, by a very small quantity; when $\theta = \frac{\pi}{2}$ the values of n are $m+2k+1$, where k is a positive integer; and when θ is very small the values of n tend to increase indefinitely. Hence, supposing the family of curves $P_n^{-m}(\cos\theta) = 0$ to be drawn, $n\ (\equiv x)$ being the abscissa, and $\theta\ (\equiv y)$ the ordinate, any one of them (say the k-th) starts from a point not on the line $y = \pi$, but indefinitely near to the point $x = m+k$ on it, bends steadily towards the right, cuts the line $y = \frac{\pi}{2}$ at the point $x = m+2k+1$, and then approaches the line $y = 0$ asymptotically, there being no point of inflexion on the curve. The following properties of the zeros of $P_n^{-m}(z)$ considered as a function of z, where n and m are real positive quantities, are immediate consequences. The number of the real values of z lying between -1 and 1 for which $P_n^{-m}(z)$ vanishes is the greatest integer less than $n-m+1$; for this is the number of the curves cut by the line $x = n$. If the number of these zeros is even, $2s$, half of them are positive and the other half negative; for the extreme curve on the right cut by $x = n$ is the one which starts from a point near to the point $x = m+2s-1$ on the line $y = \pi$, and cuts the line $y = \frac{\pi}{2}$ where $x = m+4s-1$; further the line $x = n$ cuts $y = \frac{\pi}{2}$ between $x = m+2s-1$ and $x = m+2s$; so that s curves are crossed by

* *Proceedings*, Vol. XXXI., p. 277.
† *Loc. cit.*

$x = n$ above $y = \frac{\pi}{2}$ and s below. If the number of these zeros is odd, $2s+1$, there are s positive zeros and $s+1$ negative zeros, except when $n-m$ is an odd integer, in which case $z = 0$ is a zero, and there are in addition s positive and s negative zeros. For in this case the extreme curve on the right cut by $x = n$ is the one which starts near to $x = m+2s$ on $y = \pi$ and cuts $y = \frac{\pi}{2}$ where $x = m+4s+1$, and the straight line $x = n$ cuts $y = \frac{\pi}{2}$ between $x = m+2s$ and $x = m+2s+1$, except when $n-m = 2s+1$, in which case it cuts it at the point $x = m+2s+1$; so that, when $n-m \neq 2s+1$, $s+1$ curves are crossed by the line $x = n$ above $y = \frac{\pi}{2}$, and s below, and, when $n-m = 2s+1$, s curves are crossed by $x = n$ above $y = \frac{\pi}{2}$, s below, and one on it.

Thursday, June 13th, 1901.

Dr. HOBSON, F.R.S., President, in the Chair.

Twelve members present.

After the ballot had been taken, the President announced that the following gentlemen had been elected honorary members: — Prof. Ulisse Dini, of Pisa; Prof. Georg Cantor, of Halle; and Prof. David Hilbert, of Göttingen.

Mr. Arthur William Conway, B.A. Corpus Christi College, Oxford, was elected a member of the Society.

The following papers were communicated:—

The Theory of Cauchy's Principal Values (ii.), by Mr. G. H. Hardy.

On the general Form of Three Rational Cubes whose Sum is a Cube," by Prof. Steggall.

Invariants of Curves on the same Surface, in the neighbourhood of a common Tangent Line, by Mr. T. Stuart (communicated by Dr. J. Larmor).

Dr. Macaulay made two short impromptu communications, and Lt.-Col. Allan Cunningham made an impromptu communication about Euler's *Idoneal Numbers.* If I denote one of these numbers, they have the property that, if an odd number N be expressible in *only one way* in the form $N = mx^2 + ny^2$, wherein $mn = I$ and mx^2 is prime to ny^2, then N must be either a prime or the *square of a prime.* Euler gives a list of sixty-five idoneals of which the highest is 1848, and states that there are no more under 4000. Col. Cunningham has extended the search, and finds that there are no more under 50,000; this work has been verified by the Rev. J. Cullen.

The following presents were made to the Library:—

"Educational Times," June, 1901.

"Indian Engineering," Vol. xxix., Nos. 16–20, April 20–May 18, 1901.

Durán-Loriga, Juán J.—"Charles Hermite," 8vo pamphlet; Coruña, 1901.

Lemoine, E.—

"La géométrographie dans l'espace ou stéréométrographie" (*Comptes Rendus*); Paris, 1900.

"Suite de téorèmes et de résultats concernant la géométrie du triangle," 8vo pamphlet; Paris, 1900.

"Note sur deux nouvèles décompositions des nombres entiers," 8vo pamphlet; Paris, 1900.

"Comparaison géométrografique de diverses constructions d'un même problème," 8vo pamphlet; Paris, 1900.

"Géométrografie dans l'espace," 8vo pamphlet; Paris, 1900.

"Mémoires de la section mathématique de la Société des Naturalistes de la Nouvelle-Russie," Tome xix.; Odessa, 1899.

"Publications of the United States Naval Observatory," Series 2, Vol. i., 4to; Washington, 1900. "Sun, Moon, Planets, and Miscellaneous Stars," 1894–1899.

The following exchanges were received:—

"Supplemento al Periodico di Matematica," Anno iv., Fasc. 7, Maggio 1901; Livorno.

"Proceedings of the Royal Society," Vol. lxviii., Nos. 444, 445; 1901.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxv., Hefte 5, 6; Leipzig, 1901.

"Rendiconti del Circolo Matematico di Palermo," Tomo xv., Fasc. 1, 2; 1901.

"Bulletin de la Société Mathématique de France," Tome xxix., Fasc. 2; Paris, 1901.

"Bulletin of the American Mathematical Society," Series 2, Vol. vii., No. 8, May, 1901; New York.

"Bulletin des Sciences Mathématiques," Tome xxv., Mars 1901; Paris.

"Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche," Série 3, Vol. VII., Fasc. 4, Aprile 1901; Napoli.

"Journal für die reine und angewandte Mathematik," Bd. CXXIII., Heft 3; Berlin, 1901.

"Annali di Matematica," Serie 3, Tomo V., Fasc. 3, 4; Milano, 1901.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. X., Fasc. 8–10; Roma, 1901.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahrgang XLV., Hefte 3, 4; 1900.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," 1–22; 1901.

The Theory of Cauchy's Principal Values. (Second Paper: *The use of Principal Values in some of the Double Limit Problems of the Integral Calculus.*) *By* G. H. HARDY. Read and received June 13th, 1901.

Principal Values depending on a Parameter.

1. If $f(x, \alpha)$ is a function of the two variables x, α, which for certain values of α possesses a convergent integral from $x = a$ to $x = A$,

$$I(\alpha) = \int_a^A f(x, \alpha)\, dx$$

is a function of α defined for those values of α. We may suppose a, A independent of α; for, if they depended on α, we could make the substitution

$$x = a + (A - a)\, y,$$

and so obtain an integral with the constant limits 0, 1.

We suppose further that the values of α for which $I(\alpha)$ is defined are infinite in number, and form a *closed set* S; and that α_0 is a limiting point of the set. Then the general double limit problem of the integral calculus is: *To determine the relations between*

$$I(\alpha_0) = \int_a^A f(x, \alpha_0)\, dx$$

and the limits of indetermination of $I(\alpha)$ *for* $\alpha = \alpha_0$.

It is not difficult to show that we may without loss of generality

suppose that the parameter is either *a positive integer which tends steadily to* ∞, *or a continuous variable which tends steadily to any given value.*

These problems—problems such as that of the integration of an infinite series term by term, or of differentiation under the integral sign—are well known and have been very frequently discussed. It has, however, generally been assumed that all the integrals which occur in connection with them are *unconditionally*, convergent. In this paper I shall begin a discussion of some of the corresponding problems which arise when we are considering integrals which are only *conditionally* convergent, the *principal values*, in fact, the elementary theory of which formed the subject of my first paper.* I shall begin with the case in which the parameter is integral.

Principal Values and Infinite Series.

2. Let
$$S(x) = \sum_0^\infty u_n(x)$$
be a series whose terms are functions of x, convergent, at any rate in general—*i.e.*, with the possible exception of a closed enumerable set of points—for values of x in an interval (a, A). Then $S(x)$ is *integrable term by term over* (a, A) if

$$\int_a^A S(x)\,dx = \sum_0^\infty \int_a^A u_n(x)\,dx, \tag{1}$$

or
$$\int_a^A \lim_{N=\infty} \sum_0^N u_n(x)\,dx = \lim_{N=\infty} \int_a^A \sum_0^N u_n(x)\,dx.$$

The conditions under which this equation is true have been discussed by many writers. We may refer especially to Dini, *Grundlagen*, pp. 512–530, and Osgood, "On Non-uniform Convergence, &c.," *American Jour. of Math.*, Vol. XIX.

The question with which we are concerned at present is: Under what circumstances is (1) true when some or all of the integrals which appear in it are only principal values?

3. Let us suppose, in the first place, that the interval (a, A) is finite, and that $S(x)$ is integrable term by term across any part of (a, A) which does not include a single point α $(a<\alpha<A)$. Then, however small be the positive quantity ϵ,

$$\left(\int_a^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^A\right) \sum_0^\infty u_n\,dx = \sum_0^\infty \left(\int_a^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^A\right) u_n\,dx. \tag{1}$$

* "The Elementary Theory of Cauchy's Principal Values," *Proc. L.M.S.*, Vol. XXXIV., pp. 16–40.

Let us suppose further that

$$P\int_a^A u_n(x)\,dx$$

is convergent for every value of n, and that

$$\sum_0^\infty P\int_a^A u_n(x)\,dx$$

is convergent. Then the right-hand side is

$$\sum_0^\infty P\int_a^A u_n\,dx - \sum_0^\infty P\int_{a-\epsilon}^{a+\epsilon} u_n\,dx.$$

If, finally, we suppose that the last term tends to zero with ϵ, the left-hand side of (1) will also tend to a limit, which is by definition

$$P\int_a^A \sum_0^\infty u_n(x)\,dx\,;$$

and

$$P\int_a^A \sum_0^\infty u_n\,dx = \sum_0^\infty P\int_a^A u_n\,dx. \tag{2}$$

This equation is certainly true, then, if (i.) Σu_n is integrable term by term over any part of (a, A) not including a,

$$\text{(ii.)}\quad F(x) = \Sigma P\int_a^x u_n\,dx$$

is a continuous function of x except at a, and

$$\text{(iii.)}\quad \lim_{\epsilon=0}\{F(a-\epsilon)-F(a+\epsilon)\} = 0.$$

4. We may distinguish three cases: (i.) that in which no one of the individual terms u_n becomes infinite at $x = a$, (ii.) that in which a finite number of them become infinite, and (iii.) that in which an infinite number of them do so. The first and last are the only cases of importance, as in the second case we can consider the terms which become infinite separately.

5. (i.), (ii.). In this case the symbol of the principal value on the right of (2) of § 3 is unnecessary, *i.e.*,

$$P\int_a^A \sum_0^\infty u_n\,dx = \sum_0^\infty \int_a^A u_n\,dx.$$

And we may state the conditions of § 3 as follows: that Σu_n is in-

tegrable term by term over any part of (a, A) which does not include a, and

$$F(x) = \sum_0^\infty \int_a^x u_n \, dx$$

is a continuous function of x except at a, and

$$\lim_{\epsilon=0} \{F(a-\epsilon) - F(a+\epsilon)\} = 0.$$

6. Suppose, for instance,

$$u_n = -\frac{x^n}{a^{n+1}} \quad (0 \leqq x < a)$$

$$= \frac{a^n}{x^{n+1}} \quad (a < x \leqq 1);$$

the value of u_n for $x = a$ is immaterial. Then

$$S(x) = \frac{1}{x-a} \quad (0 \leqq x \leqq 1),$$

except for $x = a$.* Also, if $0 \leqq x < a$,

$$F(x) = -\sum_0^\infty \frac{x^{n+1}}{(n+1)\, a^{n+1}} = \log\left(1 - \frac{x}{a}\right);$$

while, if $a < x \leqq 1$,

$$F(x) = -\left[\frac{x}{a}\right]_0^a + \Big[\log x\Big]_a^x - \sum_1^\infty \left\{ \left[\frac{x^{n+1}}{(n+1)\, a^{n+1}}\right]_0^a + \left[\frac{a^n}{nx^n}\right]_a^x \right\}$$

$$= \log\left(\frac{x}{a} - 1\right).$$

Thus $F(x)$ is continuous except at a, and

$$F(a-\epsilon) - F(a+\epsilon) = 0.$$

Also

$$\int_0^1 u_n \, dx = \frac{1-a^n}{n} - \frac{1}{n+1} \quad (n > 0),$$

and

$$P\int_0^1 \frac{dx}{x-a} = \log\frac{1}{a} - 1 + \sum_1^\infty \int_0^1 u_n \, dx$$

$$= \log\left(\frac{1}{a} - 1\right).$$

7. (iii.) The simplest case in which an infinite number of the terms u_n become infinite is that in which they all become infinite owing to the occurrence in all of them of the same factor

$$\Omega_\nu (x-a).$$

Let us suppose that $\quad u_n = \Omega_\nu (x-a)\, v_n,$

where v_n is a function of x which, whatever be n, has a continuous

* Here a is the a of §§ 3–5.

derivate for all values of x in question. Then, by a lemma proved in my first paper,

$$P\int_{a-\epsilon}^{a+\epsilon} u_n\,dx = \left[v_n'\right]_{a+\mu}\int_{a-\epsilon}^{a+\epsilon}(x-a)\,\Omega_\nu(x-a)\,dx,$$

where $-\epsilon \leqq \mu \leqq \epsilon$. In particular, if

$$\Omega_\nu(x-a) = \frac{1}{x-a},$$

$$P\int_{a-\epsilon}^{a+\epsilon} u_n\,dx = 2\epsilon\left[v_n'\right]_{a+\mu}.$$

Suppose now that $$|v_n'| < V_n$$

for all values of x and n in question, V_n being independent of x and ΣV_n convergent. Then the last condition of § 3 will certainly be fulfilled.

8. Let, for instance,

$$u_0 = 1, \quad u_n = \frac{2p^n\cos nx}{\cos x - \cos a} \quad (n > 0),$$

where $0 < a < \pi$, $|p| < 1$. Then

$$S(x) = \frac{1-p^2}{(\cos x - \cos a)(1-2p\cos x + p^2)},$$

and, if we may use equation (2) of § 3,

$$(1-p^2)\,P\int_0^\pi \frac{dx}{(\cos x - \cos a)(1-2p\cos x + p^2)} = 2\sum_1^\infty p^n\,P\int_0^\pi \frac{\cos nx\,dx}{\cos x - \cos a},$$

since $$P\int_0^\pi \frac{dx}{\cos x - \cos a} = 0.$$

But the left-hand is

$$\frac{1-p^2}{1-2p\cos a + p^2}\left\{2p\int_0^\pi \frac{dx}{1-2p\cos x + p^2} + P\int_0^\pi \frac{dx}{\cos x - \cos a}\right\} = \frac{2\pi p}{1+p^2-2p\cos a}$$

$$= \frac{2\pi}{\sin a}\sum_1^\infty p^n \sin na.$$

And so $$P\int_0^\pi \frac{\cos nx\,dx}{\cos x - \cos a} = \frac{\pi\sin na}{\sin a}.$$

To justify the use of § 3 we have only to observe that

$$v_n = 2p^n\cos nx\,\frac{x-a}{\cos x - \cos a},$$

and $$|v_n'|_{a+\mu} < K|p|^n,$$

where K is some quantity independent of x and of n.

9. We have so far supposed that (a, A) is finite and contains but one point a across which the series ceases to be integrable term by term in the ordinary way. No new point arises if (a, A) contains

any other finite number of such points a. This is so even if A be infinite; for all the points a can be included in a finite interval (a, A_1), and integration over (A_1, ∞) is merely an ordinary case of integration term by term.

10. Thus, for instance,

$$P\int_0^\infty \frac{1}{1-2p\cos x+p^2}\,\frac{dx}{x^2-a^2} \quad (a>0,\ |p|<1)$$

$$= \frac{1}{1-p^2}\left[P\int_0^\infty \frac{dx}{x^2-a^2} + 2\sum_1^\infty p^n P\int_0^\infty \frac{\cos nx}{x^2-a^2}\,dx\right]$$

$$= -\frac{\pi}{a(1-p^2)}\sum_1^\infty p^n \sin na = -\frac{\pi}{a(1-p^2)}\,\frac{p\sin a}{1-2p\cos a+p^2}.$$

This integral is given by De Haan (*Tables*, 193, 1).

11. Let us suppose now that there are infinitely many points a. I shall confine myself at present to the simplest case; I suppose $A=\infty$, the points a (a_i) isolated,

$$a < a_1 < a_2 < \ldots \quad (\lim_{i=\infty} a_i = \infty),$$

and

$$a_{i+1}-a_i > H \quad (i = 1, 2, \ldots),$$

where H is a positive quantity independent of i. That is to say, I suppose that the principal values with which we are dealing are of the type covered by the earlier definitions of my first paper. There is no particular difficulty in applying similar considerations to the more general cases dealt with by the later definitions.

I suppose, moreover, that the conditions of § 3 are satisfied over any finite interval (a, A_1), provided $A_1 \neq a_i$—that is to say, that Σu_n is integrable, term by term, over any part of such an interval which does not include any point a_i; that

$$F(x) = \sum_0^\infty P\int_a^x u_n\,dx$$

is a continuous function of x, except at the points a_i; and that

$$\lim_{\epsilon=0}\{F(a_i-\epsilon)-F(a_i+\epsilon)\} = 0 \quad (i = 1, 2, \ldots).$$

Then, if δ be any small positive quantity, and

$$|x-a_i| > \delta \quad (i = 1, 2, \ldots),$$

$$F(x)-F(a) = P\int_a^x \Sigma u_n\,dx. \tag{1}$$

Now let us suppose that

$$P\int_a^\infty \Sigma u_n\,dx$$

is convergent; that is, that when x tends to ∞ through any system of values satisfying the above conditions the right side of (1) tends to a finite limit independent of the particular system chosen, and therefore independent of δ. Then $F(x)$ tends to a limit, and, if

$$\lim F(x) = \Sigma P\int_a^\infty u_n\,dx,$$

it will follow that

$$P\int_a^\infty \Sigma u_n\,dx = \Sigma P\int_a^\infty u_n\,dx. \tag{2}$$

12. Let us apply this formula to the evaluation of

$$P\int_0^\infty \cos ax \cot \alpha x \frac{x\,dx}{1+x^2} \quad (0<a<2\alpha),$$

which, after my previous paper, we know to be determinate.

Since
$$\cot \alpha x = \frac{1}{\alpha x} + 2\sum_1^\infty \frac{\alpha x}{\alpha^2x^2-n^2\pi^2},$$

$$P\int_0^\infty \cos ax \cot \alpha x \frac{x\,dx}{1+x^2} = \frac{1}{\alpha}\int_0^\infty \frac{\cos ax\,dx}{1+x^2} + 2\alpha\sum_1^\infty P\int_0^\infty \frac{x^2\cos ax\,dx}{(1+x^2)(\alpha^2x^2-n^2\pi^2)},$$

if we may use the formula. Now

$$P\int_0^\infty \frac{x^2\cos ax\,dx}{(1+x^2)(\alpha^2x^2-n^2\pi^2)} = \frac{1}{\alpha^2+n^2\pi^2}\left\{\int_0^\infty \frac{\cos ax\,dx}{1+x^2} + n^2\pi^2 P\int_0^\infty \frac{\cos ax\,dx}{\alpha^2x^2-n^2\pi^2}\right\}$$

$$= \frac{1}{2}\frac{\pi}{\alpha^2+n^2\pi^2}\left\{e^{-a} - \frac{n\pi}{\alpha}\sin\frac{na\pi}{\alpha}\right\}.$$

Also
$$\sum_1^\infty \frac{1}{\alpha^2+n^2\pi^2} = \frac{1}{2\alpha}\left(\coth\alpha - \frac{1}{\alpha}\right),$$

$$\sum_1^\infty \frac{\frac{n\pi}{\alpha}\sin\frac{na\pi}{\alpha}}{\alpha^2+n^2\pi^2} = \frac{1}{2\alpha}\,\frac{\sinh(\alpha-a)}{\sinh\alpha} \quad (0<a<2\alpha).$$

Hence
$$P\int_0^\infty \cos ax \cot \alpha x \frac{x\,dx}{1+x^2} = \tfrac{1}{2}\pi\left\{e^{-a}\coth\alpha - \frac{\sinh(\alpha-a)}{\sinh\alpha}\right\} = \frac{\pi\cosh a}{e^{2\alpha}-1}.$$

This result may be found in other ways. See the *Quarterly Journal*, 1900, p. 126.

We have still to justify our use of the formula of § 11. This, as might be anticipated, requires an argument of some little complexity. In the first place it is clear, after §§ 3–8, that (1) of § 11 holds; and so what we have to prove is that we can so choose ξ that

$$\sum_1 P\int_x^\infty \frac{x^2\cos ax\,dx}{(1+x^2)(\alpha^2x^2-n^2\pi^2)}$$

is numerically less than any assigned positive quantity σ, for all values of $x>\xi$, and such that

$$|x-a_i| > \delta \quad (i = 1, 2, \ldots);$$

and that however small be δ.

Now this quantity is

$$\sum \frac{1}{a^2+n^2\pi^2}\int_x^\infty \frac{\cos ax\,dx}{1+x^2} + \sum \frac{n^2\pi^2}{a^2+n^2\pi^2}\, P\int_x^\infty \frac{\cos ax\,dx}{a^2x^2-n^2\pi^2},$$

and evidently we need only trouble about the second part.

Now suppose either $\qquad (N-\frac{1}{2})\frac{\pi}{a} < x < \frac{N\pi}{a} - \delta,$

or $\qquad \frac{N\pi}{a} + \delta < x < (N+\frac{1}{2})\frac{\pi}{a},$

say the former. Then, if $n < N$,

$$\left|\frac{n^2\pi^2}{a^2+n^2\pi^2}\int_{(N-\frac12)\pi/a}^{x}\right| < \int_{(N-\frac12)\pi/a}^{(N+\frac12)\pi/a} \frac{dx}{a^2x^2-n^2\pi^2}$$

$$< \frac{1}{2an\pi}\log\left\{\frac{N+\frac12-n}{N-\frac12-n}\,\frac{N-\frac12+n}{N+\frac12+n}\right\};$$

if $n > N$, $$< \frac{1}{2an\pi}\log\left\{\frac{n-N-\frac12}{n-N+\frac12}\,\frac{N-\frac12+n}{N+\frac12+n}\right\};$$

and, if $n = N$, $$< \frac{1}{a\delta}\,\frac{1}{(2N-\frac12)\pi}\,\frac{\pi}{2a}.$$

This last quantity can be made as small as we please by choice of N. Again,

$$\sum_1^{N-1}\frac{1}{n}\log\left\{\frac{N+\frac12-n}{N-\frac12-n}\,\frac{N-\frac12+n}{N+\frac12+n}\right\} = \sum_1^{N_1} + \sum_{N_1+1}^{N-1} \quad [N_1 = E(\sqrt{N})]$$

$$< \log\left\{\frac{N-\frac12}{N-\frac12-N_1}\,\frac{N+\frac12+N_1}{N+\frac12}\right\}$$

$$+ \frac{1}{N_1}\log\left\{\frac{N-\frac12-N_1}{\frac12}\,\frac{2N-\frac12}{N+\frac12+N_1}\right\},$$

which can be made as small as we please by choice of N. Also

$$\sum_{N+1}^{\infty}\frac{1}{n}\log\left\{\frac{n-N+\frac12}{n-N-\frac12}\,\frac{N+\frac12+n}{N-\frac12+n}\right\} = \sum_{N+1}^{2N} + \sum_{2N}^{\infty}$$

$$< \frac{1}{N+1}\log\left\{\frac{N+\frac12}{\frac12}\,\frac{3N+\frac12}{2N+\frac12}\right\} + \sum_{2N}^{\infty},$$

which too can be made as small as we please by choice of N. Hence, finally,

$$\left|\sum \frac{n^2\pi^2}{a^2+n^2\pi^2}\int_{(N-\frac12)\pi/a}^{x}\right|$$

can be made as small as we please by choice of N. And so we need only consider

$$\sum \frac{n^2\pi^2}{a^2+n^2\pi^2}\, P\int_{(N-\frac12)\pi/a} \frac{\cos ax\,dx}{a^2x^2-n^2\pi^2}.$$

We consider first the terms for which $n < N$: in them no P is needed. As x increases from $(N-\frac12)\frac{\pi}{a}$, $\cos ax$ oscillates, and $\frac{1}{a^2x^2-n^2\pi^2}$ steadily decreases. And so, if $(m-\frac12)\frac{\pi}{a}$ is the first odd multiple of $\frac{\pi}{2a}$ which is $\geqq (N-\frac12)\frac{\pi}{a}$, and $(N+p-\frac12)\frac{\pi}{a}$

is the first odd multiple of $\frac{\pi}{2a}$ which is $\geqq (m+\frac{1}{2})\frac{\pi}{a}$,

$$\left| \frac{n^2\pi^2}{a^2+n^2\pi^2} \int_{(N-\frac{1}{2})\pi/a}^{\infty} \right| < \int_{(N-\frac{1}{2})\pi/a}^{(N+p-\frac{1}{2})\pi/a} \frac{dx}{a^2x^2-n^2\pi^2}$$

$$< \frac{1}{2an\pi} \log \left\{ \frac{N+p-\frac{1}{2}-n}{N-\frac{1}{2}-n} \cdot \frac{N-\frac{1}{2}+n}{N+p-\frac{1}{2}+n} \right\};$$

and it follows by a slight modification of our previous argument that

$$\left| \sum_{1}^{N-1} \frac{n^2\pi^2}{a^2+n^2\pi^2} \int_{(N-\frac{1}{2})\pi/a}^{\infty} \right|$$

can be made as small as we please by choice of N.

There remain the terms for which $n \geqq N$. We may write them in the form

$$\tfrac{1}{2} \sum_{N}^{\infty} \frac{n\pi}{a^2+n^2\pi^2} \left(\int_{(N-\frac{1}{2})\pi/a}^{\infty} \frac{\cos ax}{ax+n\pi}\, dx + P \int_{(N-\frac{1}{2})\pi/a}^{\infty} \frac{\cos ax}{ax-n\pi}\, dx \right). \tag{1}$$

The first series is

$$\tfrac{1}{2} \sum_{N}^{\infty} \frac{n\pi}{a^2+n^2\pi^2} \sum_{N}^{\infty} \int_{(i-\frac{1}{2})\pi/a}^{(i+\frac{1}{2})\pi/a} \frac{\cos ax}{ax+n\pi}\, dx = \tfrac{1}{2} \sum_{N}^{\infty} \frac{n\pi}{a^2+n^2\pi^2} \sum_{N}^{\infty} \int_{-\pi/2a}^{\pi/2a} \frac{\cos a\left(x+\frac{i\pi}{a}\right)}{ax+(i+n)\pi}\, dx.$$

Now we may sum the series

$$\sum_{N}^{\infty} \frac{\frac{\cos}{\sin} \frac{ai\pi}{a}}{ax+(i+n)\pi}$$

under the integral sign. And it is equal, by Abel's lemma, to

$$\sum_{N}^{\infty} \left\{ \frac{1}{ax+(i+n)\pi} - \frac{1}{ax+(i+1+n)\pi} \right\} S_i,$$

where $$S_i = \sum_{N}^{i} \frac{\cos}{\sin} \frac{a\kappa\pi}{a}.$$

This is in absolute value $$< \frac{S}{ax+(N+n)\pi},$$

where S is the numerically greatest of the moduli of the sums S_i; and therefore $< \frac{S}{n\pi}$. And the first series in (1) is therefore numerically less than

$$\frac{\pi S}{a} \sum_{N}^{\infty} \frac{1}{a^2+n^2\pi^2},$$

and can be made as small as we please by choice of N.

The second series in (1) is

$$\tfrac{1}{2} \sum_{n=N}^{\infty} \frac{n\pi}{a^2+n^2\pi^2} \sum_{i=N}^{\infty} P \int_{-\pi/2a}^{\pi/2a} \frac{\cos a\left(x+\frac{i\pi}{a}\right)}{ax+(i-n)\pi}\, dx.$$

We separate the terms which correspond to one value of n into two classes, from $i=N$ to $i=2n-N$, and from $i=2n-N+1$ to $i=\infty$. By an argument similar to that which we used when we were considering the terms for which $n<N$, we can show (i.) that the series $$\tfrac{1}{2}\sum_{N}^{\infty}\left(\sum_{2n-N+1}^{\infty}\right)$$

is convergent, and (ii.) that it is numerically less than a certain constant multiple of

$$\sum_N^\infty \frac{n}{a^2+n^2\pi^2}\,\frac{1}{n-N+\frac12},$$

which is
$$< \frac{1}{\pi^2}\sum_N^\infty \frac{1}{n\,(n-N+\frac12)} < \frac{1}{N\pi^2}\left\{2+\sum_{N+1}^\infty\left(\frac{1}{n-N}-\frac{1}{n}\right)\right\}$$
$$< \frac{1}{N\pi^2}\left\{2+\sum_1^N \frac{1}{n}\right\};$$

and this can be made as small as we please by choice of N.

It only remains to consider

$$\tfrac12 \sum_N^\infty \frac{n\pi}{a^2+n^2\pi^2}\left[\sum_N^{2n-N} P\int_{-\pi/2a}^{\pi/2a} \frac{\cos a\left(x+\frac{i\pi}{a}\right)}{ax+(i-n)\pi}\,dx\right].$$

That this series is convergent follows from what precedes. Also the inner sum is

$$\left[P\int_{(N-\frac12)\pi/a}^{(2n-N+\frac12)\pi/a}\frac{\cos ax}{ax-n\pi}\,dx = P\int_{(N-n-\frac12)\pi/a}^{(n-N+\frac12)\pi/a}\frac{\cos a\,(x+n\pi)}{ax}\,dx\right.$$

$$\doteq \int_0^{(n-N+\frac12)\pi/a}\frac{\cos a\left(x+\frac{n\pi}{a}\right)-\cos a\left(x-\frac{n\pi}{a}\right)}{ax}\,dx$$

$$= -2\sin\frac{na\pi}{a}\int_0^{(n-N+\frac12)\pi/a}\frac{\sin ax}{ax}\,dx$$

$$= -2\sin\frac{na\pi}{a}\sum_0^{n-N} v_k,$$

where
$$v_k = \int_{(k-\frac12)\pi/a}^{(k+\frac12)\pi/a}\frac{\sin ax}{ax}\,dx \quad (k>0),$$
$$v_0 = \int_0^{\pi/2a}\frac{\sin ax}{ax}\,dx.$$

Now, let
$$u_n = -\frac{n\pi\sin\frac{na\pi}{a}}{a^2+n^2\pi^2}.$$

Then
$$\sum_N^\infty u_n \sum_0^{n-N} v_k = u_N v_0 + u_{N+1}(v_0+v_1)+\ldots = \sum_0^\infty v_k \sum_{N+k}^\infty u_n$$

we assume for the moment that this transformation is legitimate). Now v_k, as it is easy to see, decreases like $\frac{1}{k}$ as k increases. And

$$\sum_{N+k}^\infty u_n = \sum_{N+k}^\infty\left\{\frac{(n+1)\pi}{a^2+(n+1)^2\pi^2}-\frac{n\pi}{a^2+n^2\pi^2}\right\}S_n,$$

where
$$S_n = \sum_{N+k}^n \sin\frac{na\pi}{a},$$

and is therefore numerically less than a constant multiple of

$$\frac{(N+k)\,\pi}{a^2+(N+k)^2\pi^2};$$

and, *a fortiori*, numerically less than a constant multiple of $\frac{1}{k}$. Hence $v_k\sum_{N+k}^\infty u_n$

is numerically less than a constant multiple of $\frac{1}{k^2}$. Moreover, when k is fixed, it decreases indefinitely as N increases. It follows that we can make

$$\sum_0^\infty v_k \sum_{N+k}^\infty u_n$$

as small as we please by choice of N.—*October*, 1901.]

It only remains to show that our assumption as to the transformation of $\sum_N^\infty u_n \sum_0^{n-N} v_k$ was justified. I pass over the proof of this, as it is not difficult, and presents no point of special interest in connexion with my present subject. I conclude, finally, that *the series*

$$\sum_1^\infty \frac{n^2\pi^2}{\alpha^2+n^2\pi^2} P\int_{(N-\frac{1}{2})\pi/\alpha}^\infty \frac{\cos ax\,dx}{\alpha^2x^2-n^2\pi^2}$$

can be made as small as we please by choice of N. It follows that the use I made at the beginning of this section of the formula of § 11 was legitimate.

It was really by this method that Legendre and Lacroix "verified" Cauchy's formulæ

$$P\int_0^\infty \frac{\cos ax}{\cos bx}\frac{dx}{1+x^2} = \tfrac{1}{2}\pi\frac{\cosh a}{\cosh b} \quad (0 \leq a \leq b), \ \ldots;$$

(see their *Rapport* on his "Mémoire sur les Intégrales définies," Cauchy, *Œuvres*, Vol. I.). The preceding analysis will be sufficient to show how little they had appreciated the difficulties which it involves.*

Similarly

$$P\int_0^\infty \cos ax\cot\alpha x\,\frac{x\,dx}{1-x^2} = \frac{1}{\alpha}P\int_0^\infty \frac{\cos ax}{1-x^2}dx + 2\alpha\sum_1^\infty P\int_0^\infty \frac{x^2\cos ax\,dx}{(1-x^2)(\alpha^2x^2-n^2\pi^2)}.$$

Now
$$P\int_0^\infty \frac{x^2\cos ax\,dx}{(1-x^2)(\alpha^2x^2-n^2\pi^2)} = \frac{1}{\alpha^2-n^2\pi^2}\left\{P\int_0^\infty \frac{\cos ax\,dx}{1-x^2} + n^2\pi^2 P\int_0^\infty \frac{\cos ax\,dx}{\alpha^2x^2-n^2\pi^2}\right\}$$

$$= \tfrac{1}{2}\frac{\pi}{\alpha^2-n^2\pi^2}\left\{\sin a - \frac{n\pi}{\alpha}\sin\frac{na\pi}{\alpha}\right\}.$$

Also
$$\sum_1^\infty \frac{1}{\alpha^2-n^2\pi^2} = \frac{1}{2\alpha}\left(\cot\alpha - \frac{1}{\alpha}\right),$$

$$\sum_1^\infty \frac{\frac{n\pi}{\alpha}\sin\frac{na\pi}{\alpha}}{\alpha^2-n^2\pi^2} = -\frac{1}{2\alpha}\frac{\sin(\alpha-a)}{\sin\alpha} \quad (0 < a < 2\alpha),$$

and so
$$P\int_0^\infty \cos ax\cot\alpha x\,\frac{x\,dx}{1-x^2} = \tfrac{1}{2}\pi\left\{\sin a\cot\alpha + \frac{\sin(\alpha-a)}{\sin\alpha}\right\} = \tfrac{1}{2}\pi\cos a.$$

Principal Values containing a Continuous Parameter.

13. We shall suppose now that the parameter α is *continuous*, and, in the first place, that the range of integration (a, A) and the range of variation of the parameter α are finite.

We suppose, moreover, that the infinities of $f(x, \alpha)$ across which

* [Though I have no doubt it might be simplified to some extent.—*Nov.* 6, 1901.]

$\int f\,dx$ is not unconditionally convergent lie (for the values of x and a in question) on a finite number of continuous curves $x = X_i(a)$, which do not meet, and have at every point a definite direction nowhere parallel to x.

Uniform and Regular Convergence.

14. The principal value

$$P\int_a^A f(x, a)\,dx \tag{1}$$

will be said to be *uniformly convergent in* (β, γ) if (i.) it is convergent for every value of a in (β, γ); and (ii.) we can find a pair of positive quantities δ_0, ϵ_0 corresponding to any assigned positive quantity σ, such that

$$\left|\int_x^{x+\epsilon} f\,dx\right| < \sigma$$

for all values of a in (β, γ), every $\epsilon \leqq \epsilon_0$, and every value of x such that $a \leqq x$, $x+\epsilon \leqq A$, and x, $x+\epsilon$ differ by at least δ_0 from any of X_i; and

$$\left|P\int_{X_i-\delta}^{X_i+\delta} f\,dx\right| < \sigma$$

for all values of a in (β, γ), and every $\delta \leqq \delta_0$.

I may remark (i.) that the possibility of any of the curves $x = X_i(a)$ meeting $x = a$ or $x = A$ is excluded by the first condition, and (ii.) that the second presupposes the first.

15. Thus, if $\quad f(x, a) = \Omega_\nu(x-a)\,\Theta(x, a),$

where Θ is a function whose derivate $\dfrac{\partial\Theta}{\partial x}$ is a continuous function of both variables,

$$P\int_a^A f(x, a)\,dx$$

is uniformly convergent in $(a+\xi, A-\xi')$, if $0<\xi<\xi+\xi'<A-a$.

For, in the first place, condition (i.) is satisfied. Again

$$P\int_{a-\delta}^{a+\delta} = \Theta'_x(a+\mu, a)\int_{a-\delta}^{a+\delta}(x-a)\,\Omega_\nu(x-a)\,dx;$$

and, however small be σ, we can choose δ_0 so that the modulus of this is $<\sigma$ for all values of a in question, and every $\delta \leqq \delta_0$.

Moreover, if $a \leqq x < x+\epsilon \leqq a-\delta_0$ (or $a+\delta_0 \leqq x < x+\epsilon \leqq A$),

$$\int_x^{x+\epsilon} \Omega_\nu (x-a)\, \Theta\, dx = \int_u^{u+\epsilon} \Omega_\nu (u)\, \Theta (u+a,\, a)\, du$$

$$< K \int_u^{u+\epsilon} |\, \Omega_\nu (u)\, |\, du \text{ (where } K \text{ is a constant)}$$

$$< \frac{K}{\delta_0} \int_u^{u+\epsilon} u\, |\, \Omega_\nu (u)\, |\, du;$$

and, however small be σ, δ_0, we can choose ϵ_0 so that the modulus of this is $< \sigma$ for all values of u and a in question, and every $\epsilon \leqq \epsilon_0$. Hence condition (ii.) is satisfied.

Again, if $$f(x,\, a) = \Omega_\nu \{x - X(a)\}\, \Theta (x,\, a),$$

where Θ is a function satisfying the same conditions as before, and $X(a)$ is a function of a with a continuous and positive derivate, (1) will be uniformly convergent in (β, γ) if

$$a < X(\beta) < X(\gamma) < A.$$

This follows at once if we put $X(a) = \beta$, and treat f as a function of β.

16. Thus (i.) $P\int_a^A \frac{dx}{x-a}$, $P\int_a^A \frac{l(x-a)}{x-a}\, dx$ are uniformly convergent in $(a+\xi,\, A-\xi')$ if $0 < \xi < \xi + \xi' < A - a$.

(ii.) $P\int_0^\pi \frac{dx}{\sin (x-a)}$, $P\int_0^\pi \frac{l \sin (x-a)}{\sin (x-a)}\, dx$ are uniformly convergent in $(n\pi + \xi,\, n\pi + \pi - \xi')$, $n = 0, 1, \ldots$, if $0 < \xi < \xi + \xi' < \pi$.

(iii.) $P\int_0^{2\pi} \frac{\cos ax\, dx}{\cos x - \cos a}$ is uniformly convergent in $(n\pi + \xi,\, n\pi + \pi - \xi')$, $n = 0, 1, \ldots$, if $0 < \xi < \xi + \xi' < \pi$.

(iv.) $P\int_0^{2\pi} \frac{\cos ax\, dx}{\cos ax - \cos \theta}$ is uniformly convergent in

$$\left(\frac{2n\pi - \theta}{2\pi} + \xi,\ \frac{2n\pi + \theta}{2\pi} - \xi'\right) \quad (n = 1, 2, \ldots),$$

and in $$\left(\frac{2n\pi + \theta}{2\pi} + \xi_1,\ \frac{2n\pi + 2\pi - \theta}{2\pi} - \xi_1'\right) \quad (n = 0, 1, \ldots),$$

if $0 < \theta < \pi$, $0 < \xi < \xi + \xi' < \theta$, $0 < \xi_1 < \xi_1 + \xi_1' < \pi - \theta$.

Again, if $$f(x) = \Omega_\nu (x - X)\, \Theta (x),$$

where Θ is a function which has a continuous derivate $\frac{d\Theta}{dx}$, and $a < X < A$,

$$P\int_a^A \sin \alpha x f(x)\, dx, \quad P\int_a^A \cos \alpha x f(x)\, dx$$

are uniformly convergent in any finite interval (β, γ).

17. The principal value

$$P\int_a^A f(x, \alpha)\, dx$$

will be said to be *regularly convergent in* (β, γ) if (i.) it is uniformly convergent in any part of (β, γ) which does not include any one of a finite number of points $\alpha_1', \ldots, \alpha_r'$, for which it ceases to be determinate; and (ii.) we can find positive quantities ξ, ξ' and $p_i < p_0$, corresponding to any assigned positive quantity p_0, such that

$$P\int_{a+p_i}^{A-p_i} f(x, \alpha)\, dx$$

is uniformly convergent in

$$(\alpha_i' - \xi, \ \alpha_i' + \xi') \quad (i = 1, 2, \ldots, r).$$

If $\alpha_1' = \beta$, or $\alpha_r' = \gamma$, it is sufficient that $P\int_{a+p_i}^{A-p_i}$ be uniformly convergent in $(\beta, \beta + \xi')$ or $(\gamma - \xi, \gamma)$.

18. This case arises when the conditions for *uniform* convergence are violated owing to some of the curves $x = X_i(\alpha)$ meeting $x = a$ or $x = A$. Then $\alpha_1', \ldots, \alpha_r'$ are roots of the equations

$$a = X_i(\alpha), \quad A = X_i(\alpha).$$

Thus (i.) $P\int_a^A \frac{dx}{x-\alpha}$, $P\int_a^A \frac{l(x-\alpha)}{x-\alpha}\, dx$ are regularly convergent in any finite interval of values of α. The exceptional values of α are a, A; if $\alpha < a$, or $\alpha > A$, the integrals are unconditionally convergent.

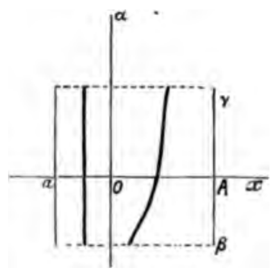
(ii.) $P\int_0^\pi \frac{dx}{\sin(x-\alpha)}$, $P\int_0^\pi \frac{l \sin(x-\alpha)}{\sin(x-\alpha)}\, dx$ are regularly convergent in any finite interval of values of α. The exceptional values of α are 0, $n\pi$.

(iii.) $P\int_0^\pi \frac{\cos ax\, dx}{\cos x - \cos \alpha}$ is regularly convergent in $(2n\pi - \pi + \xi, \ 2n\pi + \pi - \xi')$, if $0 < \xi < \xi + \xi' < 2\pi$; but not in any interval which includes any of the points $(2n+1)\pi$.

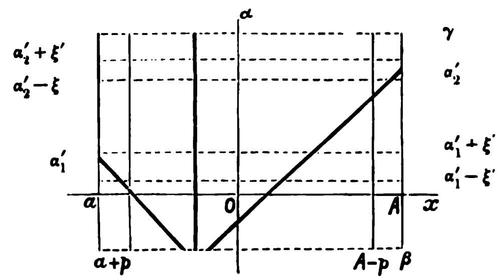
(iv.) $P\int_0^{2\pi} \frac{\cos ax\, dx}{\cos \alpha x - \cos \theta}$ $(0 < \theta < \pi)$ is regularly convergent in any finite interval of values of α. The exceptional values of α are $\frac{2n\pi \pm \theta}{2\pi}$.

A glance at the figures may make the examples of this paragraph and § 16 clearer. The thick lines are the curves $x = X_i(a)$. In (iv.) they are the rectangular hyperbolas

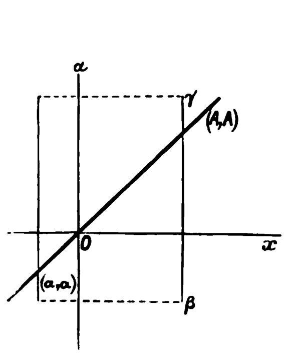
$$ax = \frac{2n\pi \pm \theta}{2\pi}.$$



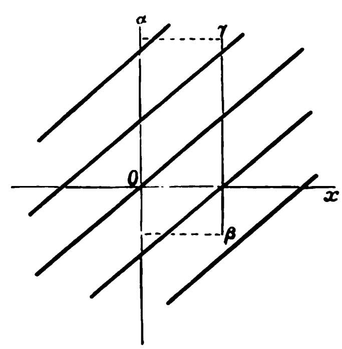
Uniform convergence, finite range.



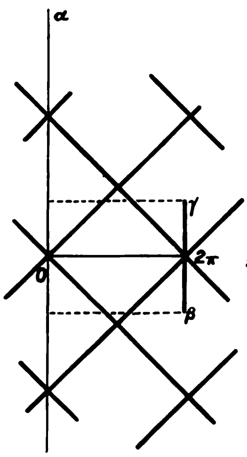
Regular convergence, finite range.



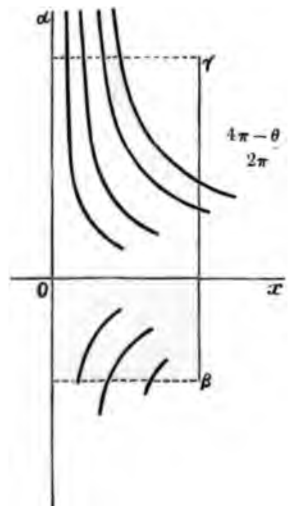
(i.) $f(x) = \dfrac{1}{x-a},\ \dfrac{l(x-a)}{x-a}.$



(ii.) $f(x) = \dfrac{1}{\sin(x-a)},\ \dfrac{l\sin(x-a)}{\sin(x-a)}.$



(iii.) $f(x) = \dfrac{\cos ax}{\cos x - \cos a}$.



(iv.) $f(x) = \dfrac{\cos ax}{\cos ax - \cos \theta}$.

Infinite Limits.

19. The principal value

$$P\int_a^\infty f(x, a)\,dx$$

will be said to be *uniformly convergent in* (β, γ) if (i.) it is convergent for every value of a in (β, γ); and (ii.) we can find a quantity A, corresponding to any assigned positive quantity σ, such that

$$P\int_a^A f(x, a)\,dx$$

is uniformly convergent in (β, γ), and

$$\left| P\int_A^\infty f(x, a)\,dx \right| < \sigma$$

for all values of a in (β, γ).*

* [It is to be observed that we do not demand that condition (ii.) should be satisfied for all values of A greater than a certain finite value, as we do in the corresponding condition for the uniform convergence of an ordinary integral. A similar remark applies to the definition of *regular* convergence in § 21.—*October*, 1901.]

20. Thus (i.) $P\int_0^\infty \frac{1}{\cos x-\cos a}\,\frac{dx}{\theta^2+x^2}$ is uniformly convergent in $(\xi,\ \pi-\xi')$ if $0<\xi<\xi+\xi'<\pi$.

(ii.) $P\int_0^\infty \frac{1}{\cos(x-a)}\,\frac{dx}{\theta^2+x^2}$ is uniformly convergent in $\left(\frac{\pi}{2}+\xi,\ \frac{3\pi}{2}-\xi'\right)$ if $0<\xi<\xi+\xi'<\pi$.

Consider (i.), for instance. In the first place, $P\int_0^{2N\pi}$ is uniformly convergent in $(\xi,\ \pi-\xi')$. Also

$$P\int_{2N\pi}^\infty = \sum_N^\infty P\int_0^{2\pi} \frac{1}{\cos x-\cos a}\,\frac{dx}{\theta^2+(x+2i\pi)^2}$$

$$= \frac{1}{2\sin a}\sum_N^\infty P\int_0^{2\pi}\left\{\cot\tfrac12(a-x)+\cot\tfrac12(a+x)\right\}\frac{dx}{\theta^2+(x+2i\pi)^2}.$$

Now
$$P\int_0^{2\pi}\cot\tfrac12(a-x)\,\frac{dx}{\theta^2+(x+2i\pi)^2} = P\int_0^{2a}+\int_{2a}^{2\pi}.$$

The second term is numerically less than

$$\frac{K}{\theta^2+(2i\pi)^2},$$

where K is a suitably chosen constant. And the first

$$= 2a\left[\cot\tfrac12(a-x)\,\frac{x-a}{\theta^2+(x+2i\pi)^2}\right]'_{a+\mu},$$

where $-a \leqq \mu \leqq a$. This, too, is numerically $<\frac{K'}{\theta^2+(2i\pi)^2}$. Finally, $\frac{1}{\sin a}$ is less than the greater of $\operatorname{cosec}\xi$, $\operatorname{cosec}\xi'$; and $\sum_N^\infty \frac{1}{\theta^2+(2i\pi)^2}$ can be made as small as we please by choice of N. Hence $P\int_{2N\pi}^\infty$ can be made $<\sigma$, by choice of N, for all values of a in $(\xi,\ \pi-\xi')$.

The uniform convergence of (ii.) may be proved in the same way. And by a slight modification of some of the arguments of my first paper we can prove general theorems as to the uniform convergence of principal values of the forms

$$P\int_0^\infty \frac{1}{\cos x-\cos a}\,\phi(x)\,dx,\quad P\int_0^\infty \frac{1}{\cos(x-a)}\,\phi(x)\,dx,\ \ldots,$$

in suitably chosen intervals. But I shall not delay over this at present.

21. The principal value

$$P\int_a^\infty f(x,a)\,dx \tag{1}$$

will be said to be *regularly convergent in* (β,γ) if (i.) it is convergent for every value of a in (β,γ);

$$\text{(ii.)}\quad P\int_a^A f(x,a)\,dx$$

is regularly convergent in (β,γ) for every finite value of $A>a$; and

(iii.) we can find (1) a value of A, (2) a division of (β, γ) into two finite sets of intervals θ, η, and (3) a set of positive quantities p_i, each corresponding to an interval η_i, and each less than some fixed quantity p_0, corresponding to any assigned positive quantity σ, and such that

$$\left| P\int_A^\infty f(x, a)\, dx \right| < \sigma$$

for all values of a in θ, and

$$\left| P\int_{A-p_i}^\infty f(x, a)\, dx \right| < \sigma$$

for all values of a in η_i.

22. We may remark that, if $a_1^A, \ldots, a_r^A$ are the exceptional values of a (§ 17) which correspond to any value of A, the intervals η will be intervals of the type $(a_i^A - \xi, a_i^A + \xi')$. The number r may increase beyond all limit with A.

The intervals θ, η_i are all to be understood as including their extremities; so that, at the point of division of θ, η_i, *both* the conditions

$$\left| P\int_A^\infty \right| < \sigma, \quad \left| P\int_{A-p_i}^\infty \right| < \sigma$$

are satisfied.

If $P\int_a^A$ is regularly, but not uniformly, convergent in (β, γ), it ceases to converge at all for certain values of a. But $P\int_a^\infty$ can only be regularly convergent in (β, γ) if it converges for all values of a in (β, γ).

23. Theorem.—*If $\psi(x, a, a)$ is a function whose derivate $\dfrac{\partial\psi}{\partial x}$ is continuous and of constant sign for all positive values of x, and all values of a, a in question, and*

$$\lim_{x=\infty} \psi(x) = 0,$$

the principal values

$$P\int_0^\infty \frac{\sin ax}{\sin ax}\psi(x)\, dx, \quad P\int_0^\infty \frac{\cos ax}{\cos ax}\psi(x)\, dx \quad (a > 0)$$

will be convergent, so long as $\dfrac{a}{a}$ is not an odd integer. They will be uniformly convergent in any interval (b, c) of values of a throughout which this condition is satisfied, if $\lim \psi(x) = 0$ *uniformly for all these*

values of a ; and they will be regularly convergent in any interval (β, γ) *of values of* α *throughout which it is satisfied, if* $\lim \psi(x) = 0$ *uniformly for all these values of* α.

The first part of this theorem was proved in my first paper. The second part requires only a very slight modification of the proof there given of the first.

There remains the third part. We consider the first of the two principal values; and we suppose, *e.g.*,

$$0 < \beta < \gamma < a.$$

In the first place
$$P\int_0^A \frac{\sin ax}{\sin \alpha x}\psi(x)\,dx$$

is regularly convergent for any finite value of A. Also

$$P\int_A^\infty = P\int_{(N-\frac{1}{2})\pi/\alpha}^\infty - \int_{(N-\frac{1}{2})\pi/\alpha}^A$$

or
$$P\int_{(N-\frac{1}{2})\pi/\alpha}^\infty + \int_A^{(N-\frac{1}{2})\pi/\alpha},$$

if $(N-\frac{1}{2})\frac{\pi}{\alpha}$ be that odd multiple of $\frac{\pi}{2\alpha}$ between which and A lies no multiple of $\frac{\pi}{\alpha}$. Now

$$P\int_{(N-\frac{1}{2})\pi/\alpha}^\infty = \frac{1}{\alpha}P\int_{(N-\frac{1}{2})\pi}^\infty \frac{\sin\frac{au}{\alpha}}{\sin u}\psi\left(\frac{u}{\alpha}\right)du;$$

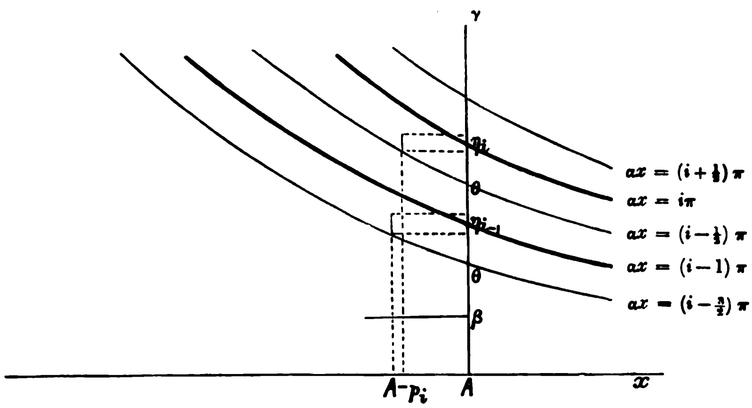
and by the second part of our theorem we can make this as small as we please by choice of N, for all values of α in question. It remains to consider $\int_{(N-\frac{1}{2})\pi/\alpha}^A$ or $\int_A^{(N-\frac{1}{2})\pi/\alpha}$.

Suppose
$$\beta < \frac{k\pi}{A} < \ldots < \frac{(k+l)\pi}{A} < \gamma.$$

The values α_i^A are $\frac{i\pi}{A}$, $i = k, \ldots, k+l$; and the intervals η_i are of the type
$$\left(\frac{i\pi}{A} - \xi,\ \frac{i\pi}{A} + \xi'\right).$$

We take $N = k$, $\xi = \xi' = \frac{\pi}{8A}$ (see the figure). And $A - p_i =$ the

value of x where $\alpha = (i-\frac{1}{8})\frac{\pi}{A}$ meets $\alpha x = (i-\frac{1}{2})\pi$; *i.e.*, $p_i = \dfrac{\frac{3}{8}A}{i-\frac{1}{8}}$.



Regular convergence, infinite range.

As $A < \dfrac{k\pi}{\beta}$, and $i \geqq k$, p_i is certainly less than $p_0 = \dfrac{\pi}{2\beta}$. Also, if α lies in η_i, and x between $(i-\frac{1}{2})\dfrac{\pi}{\alpha}$ and $A-p_i$, αx lies between $(i-\frac{1}{2})\pi$ and

$$(A-p_i)(i+\tfrac{1}{8})\frac{\pi}{A} = \frac{(i-\frac{1}{2})(i+\frac{1}{8})}{i-\frac{1}{8}}\pi;$$

so that
$$\tfrac{1}{2}\pi \geqq i\pi - \alpha x \geqq \frac{\frac{1}{4}i+\frac{1}{16}}{i-\frac{1}{8}}\pi > \tfrac{1}{4}\pi.$$

Hence
$$\left|\int_{(i-\frac{1}{2})\pi/\alpha}^{A-p_i}\right| < \sqrt{2}\,\psi'\left\{A-p_i-(i-\tfrac{1}{2})\frac{\pi}{\alpha}\right\},$$

where ψ' is the greatest value of $|\psi(x)|$ in the range of integration. The least value of x in the range is $\geqq \dfrac{i-\frac{1}{2}}{i+\frac{1}{8}}A$, and $\psi(x)$ tends to zero for $x=\infty$, uniformly for all values of α. And

$$A-p_i-(i-\tfrac{1}{2})\frac{\pi}{\alpha} \leqq \left(\frac{1}{i-\frac{1}{8}} - \frac{1}{i+\frac{1}{8}}\right)(i-\tfrac{1}{2})A < \frac{A}{4i},$$

which does not increase indefinitely with A and i. Hence we can choose A so great that

$$\left|P\int_{A-p_i}^{\infty}\right| < \sigma$$

throughout the intervals η_i.

The intervals θ are of the type

$$(i-\tfrac{7}{8})\frac{\pi}{A}, \quad (i-\tfrac{1}{8})\frac{\pi}{A};$$

and if a lies in this interval, and x between A and $(i-\frac{1}{2})\frac{\pi}{a}$, ax lies between $(i-\frac{7}{8})\pi$ and $(i-\frac{1}{8})\pi$; so that

$$\left|\frac{1}{\sin ax}\right| \leq \operatorname{cosec}\tfrac{1}{8}\pi.$$

Hence $\left|\int_A^{(i-\frac{1}{2})\pi/a}\right|$ or $\left|\int_{(i-\frac{1}{2})\pi/a}^{A}\right| < \operatorname{cosec}\frac{1}{8}\pi\,\psi'\left|(i-\frac{1}{2})\frac{\pi}{a}-A\right|$,

where ψ' is again the greatest value of $|\psi(x)|$ in the range of integration. And it follows, as before, that we can choose A so great that

$$\left|P\int_A^\infty\right| < \sigma$$

throughout the intervals θ. Hence

$$P\int_0^\infty \frac{\sin ax}{\sin ax}\psi(x)\,dx$$

is regularly convergent.

We may, for instance, suppose

$$\psi(x) = x^{-\mu}\ (0<\mu<1), \quad \frac{1}{x^2\pm\theta^2}, \quad e^{-\lambda x}\ (\lambda>0), \quad \ldots.$$

This theorem may be extended in various ways. We may suppose, *e.g.*, that $\dfrac{\partial\psi}{\partial x}$ is of constant sign only after some finite value of x independent of a and a; or we may substitute for

$$\frac{\sin ax}{\sin ax}, \quad \frac{\cos ax}{\cos ax}$$

such factors as $\sin ax \tan ax$, $\sin ax \cot ax$, ...

In these two cases the exceptional values of $\dfrac{a}{a}$ will be the *even* integral values.

It is to be observed that no difficulty arises with these exceptional values, if $\int^\infty \psi(x)\,dx$ is convergent. Thus, if in

$$P\int_0^\infty \frac{\sin ax}{\sin ax}\psi(x)\,dx$$

we make $a = \alpha$, 3α, we obtain

$$\int_0^\infty \psi(x)\,dx, \quad \int_0^\infty (3-4\sin^2 \alpha x)\,\psi(x)\,dx;$$

and the latter of these converges or diverges with the former.

24. It is to be observed that a very simple transformation may change regular into uniform convergence, or *vice versa*. Thus the substitution $\alpha x = y$ transforms the principal values of the theorem into

$$\frac{1}{\alpha} P \int_0^\infty \frac{\sin \frac{a}{\alpha} y}{\sin y}\, \psi\left(\frac{y}{\alpha}\right) dy, \quad \frac{1}{\alpha} P \int_0^\infty \frac{\cos \frac{a}{\alpha} y}{\cos y}\, \psi\left(\frac{y}{\alpha}\right) dy,$$

which are uniformly convergent in the interval of values of α in question.

Continuity of Principal Values.

25. THEOREM 1.—*If $f(x, \alpha)$ is a continuous function of both variables in any finite part of the rectangle*

$$(a, A, \beta, \gamma)$$

which does not include any point of any of the curves $x = X_s(\alpha)$, and

$$P \int_a^A f(x, \alpha)\,dx$$

is uniformly convergent in (β, γ), it will be a continuous function of α in (β, γ).

This is true whether A be finite or infinite.

In the first place, suppose A finite. We may without loss of generality suppose that there is only one curve $x = X(\alpha)$; for we can reduce any case to this by dividing the range of integration and the interval (β, γ) into a finite number of parts.

We draw two auxiliary curves

$$x = X(\alpha) \pm \delta;$$

in the region R_δ exterior to these curves $f(x, \alpha)$ is a continuous function of both variables, and therefore a uniformly continuous function of α. Let α_0 be any value of α in (β, γ); and suppose,

e.g., that $X'(a_0)>0$. Then, if h be a small positive quantity, and $a_0+h<\gamma$,

$$P\int_a^A f(x,\, a_0+h)\,dx - P\int_a^A f(x,\, a_0)\,dx$$

$$= \left(\int_a^{X(a_0)-\delta} + \int_{X(a_0+h)+\delta}^A\right)\left[f(x,\, a_0+h)-f(x,\, a_0)\right]dx$$

$$+\int_{X(a_0)-\delta}^{X(a_0+h)-\delta} f(x,\, a_0+h)\,dx - \int_{X(a_0)+\delta}^{X(a_0+h)+\delta} f(x,\, a_0)\,dx$$

$$+P\int_{X(a_0+h)-\delta}^{X(a_0+h)+\delta} f(x,\, a_0+h)\,dx - P\int_{X(a_0)-\delta}^{X(a_0)+\delta} f(x,\, a_0)\,dx.$$

Now let σ be any positive quantity. We can choose δ so small that

$$\left|P\int_{X(a)-\delta}^{X(a)+\delta}\right| < \tfrac{1}{6}\sigma$$

for all values of a in $(\beta,\, \gamma)$, and ϵ_0 so small that

$$\left|\int_x^{x+\epsilon}\right| < \tfrac{1}{6}\sigma$$

for all values of $\epsilon \leqq \epsilon_0$ and all values of x, a such that x, $x+\epsilon$ fall within R_δ. Then we can choose h' so small that

$$X(a_0+h)-X(a_0) \leqq \epsilon_0,$$

and $$|f(x,\, a_0+h)-f(x,\, a_0)| < \frac{\sigma}{3(A-a)},$$

for all values of x in either of the intervals

$$a,\ X(a_0)-\delta;\quad X(a_0+h)+\delta,\ A;$$

and all values of $h \leqq h'$. And then

$$\left|P\int_a^A f(x,\, a_0+h)\,dx - P\int_a^A f(x,\, a_0)\,dx\right| < \sigma$$

for all values of $h \leqq h'$. A similar proof applies to negative values of h, if $a_0 > \beta$. Hence $P\int_a^A$ is continuous at a_0.

In the second place, let us suppose that the upper limit is ∞. We can choose A so that $P\int_a^A$ is uniformly convergent in $(\beta,\, \gamma)$, and

$$\left|P\int_A^\infty\right| < \tfrac{1}{3}\sigma$$

[redacted]

for all values of a in (β, γ). And, since, by the first part of the theorem, $P\int_a^A$ is continuous in (β, γ), we can choose h' so small that

$$\left| P\int_a^A f(x, a_0+h)\, dx - P\int_a^A f(x, a_0)\, dx \right| < \tfrac{1}{3}\sigma$$

if $h \leqq h'$. Hence

$$\left| P\int_a^\infty f(x, a_0+h)\, dx - P\int_a^\infty f(x, a_0)\, dx \right| < \sigma$$

if $h \leqq h'$. A similar proof applies to negative values of h. Hence $P\int_a^\infty$ is continuous at a_0.

26. Thus the principal values

$$P\int_0^A \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{A-a}{A+a}, \tag{i.}$$

$$P\int_0^\infty \frac{dx}{x^2-a^2} = 0, \tag{ii.}$$

$$P\int_0^\infty \frac{\cos ax}{x^2-a^2}\, dx = -\frac{\pi}{2a}\sin a a \tag{iii.}$$

are continuous in (β, γ) if β, γ be any positive quantities. As a approaches zero they tend to the finite limits

$$-\frac{1}{A},\quad 0,\quad -\frac{a\pi}{2};$$

but they are meaningless for $a = 0$. And

$$P\int_0^\pi \frac{\cos nx\, dx}{\cos x - \cos a} = \pi\frac{\sin na}{\sin a} \tag{iv.}$$

if n be a positive integer; and this is continuous in $(\xi, \pi-\xi')$ if $0<\xi<\xi+\xi'<\pi$; and tends, as a approaches 0 or π, to the finite limits

$$n\pi,\quad (-)^{n-1} n\pi,$$

but is meaningless for $a = 0$ or π.

Again, the principal value

$$P\int_0^\infty \frac{\log(1+2a\cos ax+a^2)}{1-x^2}\, dx = \pi\tan^{-1}\frac{a\sin a}{1+a\cos a} \tag{v.}$$

is uniformly convergent in $(-1, 1)$; and, for $a = 1, -1$ becomes

$$P\int_0^\infty \frac{\log 4\cos^2\frac{1}{2}ax}{1-x^2} = \tfrac{1}{2}\pi a,$$

$$P\int_0^\infty \frac{\log 4\sin^2\frac{1}{2}ax}{1-x^2} = \tfrac{1}{2}\pi(a-\pi).$$

27. An interesting case is that in which an unconditionally convergent integral changes continuously into a principal value for some special value of a parameter.

Let us consider, for instance, the integral

$$\int_0^{2\pi} \frac{\sin x}{1+2a\cos x+a^2} f(x)\,dx, \tag{i.}$$

where $f(x)$ is a function which has a continuous derivate for all values of x in question, and a is positive and <1. Then

$$\int_x^{x+\epsilon} = \left[-\frac{1}{2a}\log(1+2a\cos x+a^2)f(x)\right]_x^{x+\epsilon} + \frac{1}{2a}\int_x^{x+\epsilon} \log(1+2a\cos x+a^2)f'(x)\,dx.$$

Now $$\frac{\partial}{\partial a}\log\frac{(1+a)^2}{1+2a\cos x+a^2} = 2\frac{1-a}{1+a}\,\frac{1-\cos x}{1+2a\cos x+a^2} \geqq 0$$

if $0 \leqq a < 1$; and so

$$0 < \log\frac{(1+a)^2}{1+2a\cos x+a^2} < \log\sec^2\tfrac{1}{2}x.$$

Hence $$\left|\frac{1}{2a}\int_x^{x+\epsilon}\log(1+2a\cos x+x^2)f'(x)\,dx\right|$$
$$< \frac{\log(1+a)}{a}\int_x^{x+\epsilon}|f'(x)|\,dx + \frac{1}{2a}\int_x^{x+\epsilon}\log\sec^2\tfrac{1}{2}x\,|f'(x)|\,dx;$$

and this can be made as small as we please, by choice of ϵ, for all values of a and x in question.

Again, $$\left|\left[-\frac{1}{2a}\log(1+2a\cos x+a^2)f(x)\right]_x^{x+\epsilon}\right|$$

can be made as small as we please, by choice of ϵ, for all values of a in $(0, 1)$, and all values of x in $(0, \pi-\delta)$ or $(\pi+\delta, 2\pi)$, where δ is any positive quantity $<\pi$, however small. But, if $x = \pi$, this condition cannot be satisfied. If, for instance, $f(x) = 1$,

$$\frac{1}{2a}\left\{\log(1-a)^2 - \log(1-2a\cos\epsilon+a^2)\right\} = \frac{1}{2a}\log\left\{1+\frac{4a\sin^2\frac{1}{2}\epsilon}{(1-a)^2}\right\},$$

and the value of ϵ which we have to take decreases beyond all limit as a approaches unity. And, in fact, since

$$\lim_{a=1}\frac{\sin x}{1+2a\cos x+a^2} = \tfrac{1}{2}\tan\tfrac{1}{2}x,$$

the integral (i.) is not convergent when $a = 1$. However,

$$P\int_0^{2\pi}\tfrac{1}{2}\tan\tfrac{1}{2}x\,f(x)\,dx$$

is convergent. Moreover,

$$\int_{\pi-\delta}^{\pi+\delta}\frac{\sin x}{1+2a\cos x+a^2}f(x)\,dx$$
$$= -\frac{1}{2a}\{f(\pi+\delta)-f(\pi-\delta)\}\log(1-2a\cos\delta+a^2) + \frac{1}{2a}\int_{\pi-\delta}^{\pi+\delta}\log(1+2a\cos x+a^2)f'(x)\,dx.$$

As before, the second term can be made as small as we please, by choice of δ, for all values of a in question. And so can the first, as it is equal to

$$-\frac{1}{2a}\log\left\{(1-a)^2+4a\sin^2\tfrac{1}{2}\delta\right\}2\delta f'(\pi+\theta\delta) \quad (-1\leqq\theta\leqq 1).$$

We can choose δ, then, so small that

$$\left|\int_{\pi-\delta}^{\pi+\delta}\frac{\sin x}{1+2a\cos x+a^2}f(x)\,dx\right| < \sigma$$

for all values of a between 0 and 1, and

$$\left| P\int_{\pi-\delta}^{\pi+\delta} \tfrac{1}{2}\tan\tfrac{1}{2}x\, f(x)\, dx \right| < \sigma.$$

Hence
$$P\int_0^{2\pi} \frac{\sin x}{1+2a\cos x+a^2} f(x)\, dx$$

is uniformly convergent in (0, 1), and therefore a continuous function of a in that interval. Consequently,

$$P\int_0^{2\pi} \tfrac{1}{2}\tan\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1}\int_0^{2\pi} \frac{\sin x}{1+2a\cos x+a^2} f(x)\, dx. \tag{1}$$

Similarly,
$$P\int_{-\pi}^{\pi} \tfrac{1}{2}\cot\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1}\int_{-\pi}^{\pi} \frac{\sin x}{1-2a\cos x+a^2} f(x)\, dx. \tag{2}$$

$$P\int_0^{2\pi} \tfrac{1}{2}\sec\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1}\int_0^{2\pi} \frac{(1+a)\sqrt{a}\cos\frac{1}{2}x}{1+2a\cos x+a^2} f(x)\, dx, \tag{3}$$

$$P\int_{-\pi}^{\pi} \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1}\int_{-\pi}^{\pi} \frac{(1+a)\sqrt{a}\sin\frac{1}{2}x}{1-2a\cos x+a^2} f(x)\, dx. \tag{4}$$

In (1) and (3) we may substitute a, A for 0, 2π as limits, provided $-\pi < a < \pi$, $\pi < A < 3\pi$. Similarly for (2) and (4).

28. We may expand the functions under the integral signs on the right in powers of a, and integrate term by term. Thus from (1) we deduce

$$P\int_0^{2\pi} \tfrac{1}{2}\tan\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1}\sum_1^\infty (-)^{n-1} a^n \int_0^{2\pi} \sin nx\, f(x)\, dx. \tag{1}$$

This is equal to
$$\sum_1^\infty (-)^{n-1}\int_0^{2\pi} \sin nx\, f(x)\, dx,$$

if the latter series is convergent. Thus, if $f(x)$ can be expanded as a Fourier series,

$$a_0 + \sum_1^\infty (a_n\cos nx + b_n \sin nx) \quad (0 < x < 2\pi),$$

$$P\int_0^{2\pi} \tfrac{1}{2}\tan\tfrac{1}{2}x\, f(x)\, dx = \pi\lim_{a=1}\sum_1^\infty (-)^{n-1} a_n a^n = \sum_1^\infty (-)^{n-1} a_n,$$

if this is convergent.

For instance,
$$\sum_1^\infty \frac{\sin nx}{n} = \tfrac{1}{2}(\pi - x) \quad (0 < x < 2\pi);$$

and therefore
$$\int_0^{2\pi} \tfrac{1}{2}\tan\tfrac{1}{2}x \,.\, \tfrac{1}{2}(\pi - x)\, dx = \pi\sum_1^\infty (-)^{n-1}\frac{1}{n} = \pi\log 2,$$

$$P\int_0^{\pi} \phi\tan\phi\, d\phi = -\pi\log 2.$$

Similarly, we deduce from (2), (3), and (4) of § 27

$$P\int_{-\pi}^{\pi} \tfrac{1}{2}\cot\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1}\sum_1^\infty a^n \int_{-\pi}^{\pi} \sin nx\, f(x)\, dx, \tag{2}$$

$$P\int_0^{2\pi} \tfrac{1}{2}\sec\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1}\sum_0^\infty (-)^n a^{n+\frac{1}{2}} \int_0^{2\pi} \cos(n+\tfrac{1}{2})\, x\, f(x)\, dx, \tag{3}$$

$$P\int_{-\pi}^{\pi} \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1}\sum_0^\infty a^{n+\frac{1}{2}} \int_{-\pi}^{\pi} \sin(n+\tfrac{1}{2})\, x\, f(x)\, dx. \tag{4}$$

In each of the series on the right we may put $a = 1$ if the resulting series are convergent.

29. In the same way we can establish four more general formulæ of which

$$P\int_0^{2\pi} \tfrac{1}{2}\tan\tfrac{1}{2}(x-\theta)\,f(x)\,dx = \lim_{a=1}\sum_1^{\infty}(-)^{n-1}a^n\int_0^{2\pi}\sin n(x-\theta)\,f(x)\,dx$$

is typical. If, for instance,

$$f(x) = \cos px,\ \sin px\ \ (p \text{ an integer}),\quad x = 2\phi,\quad \theta = 2\psi,$$

we obtain

$$P\int_0^{\pi}\tan(\phi-\psi)\cos 2p\phi\,d\phi = (-)^p\,\pi\sin 2p\psi,$$

$$P\int_0^{\pi}\tan(\phi-\psi)\sin 2p\phi\,d\phi = (-)^{p-1}\,\pi\cos 2p\psi.$$

30. Again, $P\displaystyle\int_0^{\infty}\tfrac{1}{2}\tan\tfrac{1}{2}x\,f(x)\,dx = \lim_{a=1}\sum_1^{\infty}(-)^{n-1}a^n\int_0^{\infty}\sin nx\,f(x)\,dx,$ (1)

if

$$P\int_0^{\infty}\frac{\sin x}{1+2a\cos x+a^2}\,f(x)\,dx$$

be uniformly convergent in $(0, 1)$. Now it follows from what precedes that, if $f'(x)$ is continuous, $P\int_0^{2n\pi}$ is uniformly convergent for any value of n. Hence $P\int_0^{\infty}$ will be so if we can so choose n that

$$\left|\,P\int_{2n\pi}^{\infty}\frac{\sin x}{1+2a\cos x+a^2}\,f(x)\,dx\,\right| < \sigma \tag{a}$$

for all values of a in $(0, 1)$.

Let us suppose, in the first place, that $f(x)$ is positive and tends steadily to zero for $x = \infty$. Then

$$\int_{2n\pi}^{\infty}\frac{\sin x}{1+2a\cos x+a^2}\,f(x)\,dx = \frac{1}{2a}\log(1+a)^2 f(2n\pi) + \frac{1}{2a}\int_{2n\pi}^{\infty}\log(1+2a\cos x+a^2)\,f'(x)\,dx$$

$$= \frac{1}{2a}\int_{2n\pi}^{\infty}\log\frac{1+2a\cos x+a^2}{(1+a)^2}\,f'(x)\,dx.$$

Also

$$0 < \log\frac{1+2a\cos x+a^2}{(1+a)^2}\,f'(x) < \log\cos^2\tfrac{1}{2}x\,f'(x),$$

and

$$\int_{2n\pi}^{\infty}\log\cos^2\tfrac{1}{2}x\,f'(x)\,dx$$

is convergent. Hence condition (a) can be satisfied.

If, *e.g.*,

$$f(x) = \frac{1}{x},$$

$$P\int_0^{\infty}\tfrac{1}{2}\tan\tfrac{1}{2}x\,\frac{dx}{x} = \lim_{a=1}\sum_1^{\infty}(-)^{n-1}a^n\int_0^{\infty}\sin nx\,\frac{dx}{x} = \lim_{a=1}\frac{\pi}{2(1+a)},$$

$$P\int_0^{\infty}\frac{\tan x}{x}\,dx = \tfrac{1}{2}\pi.$$

Similarly, $$P\int_0^\infty \tfrac{1}{2}\sec\tfrac{1}{2}x\, f(x)\,dx = \lim_{a=1}\sum_0^\infty (-)^n a^{n+\frac{1}{2}}\int_0^\infty \cos(n+\tfrac{1}{2})\,x\,f(x)\,dx$$
$$= \sum_0^\infty (-)^n \int_0^\infty \cos(n+\tfrac{1}{2})\,x\,f(x)\,dx,$$
if this series be convergent.

Legendre determined
$$P\int_0^\infty \frac{x\tan ax\,dx}{x^2+m^2},\quad P\int_0^\infty \frac{x\cot ax\,dx}{x^2+m^2},\quad P\int_0^\infty \frac{x\operatorname{cosec} ax\,dx}{x^2+m^2}$$
(considering them as ordinary integrals) by assuming that they were the limiting values of
$$\int_0^\infty \frac{\sin 2ax}{1+2a\cos 2ax+a^2}\,\frac{x\,dx}{x^2+m^2} = \tfrac{1}{2}\,\frac{\pi}{e^{2am}+a},$$
$$\int_0^\infty \frac{\sin 2ax}{1-2a\cos 2ax+a^2}\,\frac{x\,dx}{x^2+m^2} = \tfrac{1}{2}\,\frac{\pi}{e^{2am}-a},$$
$$\int_0^\infty \frac{\sin ax}{1-2a\cos 2ax+a^2}\,\frac{x\,dx}{x^2+m^2} = \frac{\pi}{2(1+a)}\,\frac{e^{am}}{e^{2am}-a}$$
for $a = 1$. We can see now that his assumption was correct.

31. Suppose that, in (2) of § 30,
$$f(x) = x^{-a} \quad (0 < a < 1).$$
Then $$\int_0^\infty \cos(n+\tfrac{1}{2})\,x\,\frac{dx}{x^a} = \frac{\pi}{2\Gamma(a)\cos\frac{1}{2}a\pi}\,\frac{1}{(n+\frac{1}{2})^{1-a}}.$$
Also $$\tfrac{1}{2}\sec\tfrac{1}{2}x = -2\pi\sum_0^\infty (-)^n \frac{2n+1}{x^2-(2n+1)^2\pi^2};$$
and $$\int_0^\infty \tfrac{1}{2}\sec\tfrac{1}{2}x\,\frac{dx}{x} = -2\pi\sum_0^\infty (-)^n(2n+1)\,P\int_0^\infty \frac{1}{x^2-(2n+1)^2\pi^2}\,\frac{dx}{x^a}$$
$$= -2\pi^{-a}\sum_0^\infty \frac{(-)^n}{(2n+1)^a}\,P\int_0^\infty \frac{1}{x^2-1}\,\frac{dx}{x^a}$$
$$= \pi^{1-a}\tan\tfrac{1}{2}a\pi\sum_0^\infty \frac{(-)^n}{(2n+1)^a},$$
by §§ 3–12. This series is therefore equal to
$$\frac{\pi}{2\Gamma(a)\cos\frac{1}{2}a\pi}\sum_0^\infty \frac{(-)^n}{(n+\frac{1}{2})^{1-a}}.$$
That is to say, the function $\psi(a) = \sum_0^\infty \frac{(-)^n}{(2n+1)^a} \quad (0<a<1)$

satisfies the functional equation
$$\psi(1-a) = \left(\frac{2}{\pi}\right)^a \sin\tfrac{1}{2}a\pi\,\Gamma(a)\,\psi(a).$$
This is a well known relation first proved by Schlömilch, and closely connected with the theory of Riemann's ζ-function.

32. The formulæ of § 30 may be generalized, as those of § 28 were in § 29. There will be no difficulty in proving, for example, that
$$P\int_0^\infty \frac{1}{\cos\delta x-\cos\phi}\,f(x)\,dx = \frac{2}{\sin\phi}\sum_0^\infty \sin n\phi\int_0^\infty \cos n\delta x\,f(x)\,dx,$$
if this series is convergent.

If, for instance, $$f(x) = \frac{1}{\cosh \frac{1}{2}\pi x},$$

$$\int_0^{\infty} \cos n\delta x\, f(x)\, dx = \frac{1}{\cosh n\delta}.$$

But

$$P\int_0^{\infty} \frac{1}{\cos \delta x - \cos \phi}\, \frac{dx}{\cosh \frac{1}{2}\pi x} = \frac{4}{\pi} \sum_0^{\infty} (-)^m (2m+1)\, P\int_0^{\infty} \frac{1}{\cos \delta x - \cos \phi}\, \frac{dx}{x^2 + (2m+1)^2}$$

$$= 2 \sum_0^{\infty} \frac{(-)^m}{\cosh (2m+1)\,\delta - \cos \phi}.$$

Hence $$\sum_0^{\infty} \frac{(-)^m}{\cosh (2m+1)\,\delta - \cos \phi} = \frac{1}{\sin \phi} \sum_0^{\infty} \frac{\sin n\phi}{\cosh n\delta}.$$

This becomes obvious if $\phi = \frac{1}{2}\pi$. It is really a formula in elliptic functions; for, if we write q for $e^{-\delta}$, it takes the form

$$\sum_0^{\infty} \frac{(-)^m q^{2m+1}}{1 - 2q^{2m+1}\cos \phi + q^{4m+2}} = \frac{1}{\sin \phi} \sum_0^{\infty} \frac{q^n \sin n\phi}{1 + q^{2n}}.$$

If we integrate this from $\phi = 0$ to $\phi = \pi$, observing that $\int_0^{\pi} \frac{\sin n\phi}{\sin \phi}\, d\phi$ is π or 0, according as n is odd or even, we obtain the well known formula

$$\frac{q}{1+q^2} + \frac{q^3}{1+q^6} + \dots = \frac{q}{1-q^2} - \frac{q^3}{1-q^6} + \dots$$

(Jacobi, *Fundamenta Nova*, XL. 5).

If we had taken $$\phi = \frac{1}{2}\pi, \quad (x) = \frac{1}{\cosh x - \cos \theta},$$
we should have found that

$$\frac{1}{\cosh \delta\theta} + \sum_0^{\infty} \left\{ \frac{1}{\cosh (2n\pi + \theta)\,\delta} - \frac{1}{\cosh (2n\pi - \theta)\,\delta} \right\} = \frac{2}{\sin \theta} \sum_0^{\infty} (-)^m \frac{\sinh (2m+1)(\pi - \theta)\,\delta}{\sinh (2m+1)\,\pi\delta}.$$

This too becomes obvious if $\theta = \frac{1}{2}\pi$. It is not difficult to obtain general formulæ which include these as particular cases; but my present purpose is only to show how the methods of the preceding sections can be applied to obtain results of interest in different branches of analysis.

33. The equations of § 30 also hold (except for certain exceptional values of a) if

$$f(x) = {\cos \atop \sin}\, ax\, \psi(x),$$

where $\psi(x)$ is a function whose first two derivates are continuous and of constant sign after a certain value of x, and

$$\lim_{x=\infty} \psi(x) = 0.$$

For $$\int_{2n\pi}^{\infty} \frac{\sin x}{1 + 2a\cos x + a^2} f(x)\, dx = \sum_n^{\infty} \int_{2i\pi}^{2(i+1)\pi} = \int_0^{2\pi} \frac{\sin x}{1 + 2a\cos x + a^2} F(x)\, dx,$$

where $$F(x) = \sum_n^{\infty} {\cos \atop \sin}\, a\,(x + 2i\pi)\, \psi(x + 2i\pi).$$

Now, provided a be not an integer, the series

$$\Sigma {\cos \atop \sin}\, 2ia\pi\, \psi(x + 2i\pi), \quad \Sigma {\cos \atop \sin}\, 2ia\pi\, \psi'(x + 2i\pi)$$

are uniformly convergent in $(0, 2\pi)$. It follows (i.) that, whatever be the value of n, $F(x)$ and $F'(x)$ are continuous in $(0, 2\pi)$; (ii.) that we can choose n so great that the moduli of $F(x)$, $F'(x)$ are as small as we please for all values of x in $(0, 2\pi)$.

Moreover, $\displaystyle\int_0^{2\pi} \frac{\sin x}{1+2a\cos x+a^2}\, F(x)\, dx = -\frac{1}{2a}\int_0^{2\pi} \log \frac{(1+a)^2}{1+2a\cos x+a^2}\, F'(x)\, dx,$

and
$$0 < \log\frac{(1+a)^2}{1+2a\cos x+a^2} < \log \sec^2 \tfrac{1}{2}x.$$

Hence
$$\left|\int_{2n\pi}^{\infty} \frac{\sin x}{1+2a\cos x+a^2}\, f(x)\, dx\right| < \frac{1}{2a}\int_0^{2\pi} \log\sec^2\tfrac{1}{2}x \,|\, F'(x)\,|\, dx,$$

and can therefore be made as small as we please, by choice of n, for all values of a in $(0, 1)$. Hence
$$P\int_0^{\infty} \frac{\sin x}{1+2a\cos x+a^2}\, f(x)\, dx$$
is uniformly convergent in $(0, 1)$.

Suppose, *e.g.*, that
$$f(x) = \frac{x\cos ax}{x^2+\theta^2} \quad (0<a<1).$$

Then
$$P\int_0^{\infty} \tan\tfrac{1}{2}x \cos ax \frac{x\,dx}{x^2+\theta^2} = 2\lim_{a=1}\sum_1^{\infty} (-)^{n-1} a^n \int_0^{\infty} \sin nx \cos ax \frac{x\,dx}{x^2+\theta^2}$$
$$= \tfrac{1}{2}\pi \sum_1^{\infty} (-)^{n-1} \left\{e^{-(n-a)\theta} + e^{-(n+a)\theta}\right\} = \frac{\pi\cosh a\theta}{e^{\theta}+1}.$$

Similarly,
$$P\int_0^{\infty} \cot\tfrac{1}{2}x \cos ax \frac{x\,dx}{x^2+\theta^2} = \frac{\pi\cosh a\theta}{e^{\theta}-1}.$$

This agrees with the result found in another way in § 12. A third proof will be found in the *Quarterly Journal*, No. 125, 1900, p. 120.

It is not difficult to prove that
$$P\int_0^{\infty} \frac{\sin x}{1+2a\cos x+a^2}\, f(x)\, dx$$
is still continuous in $(0, 1)$ if the conditions of § 30 are satisfied, except that $f(x)$ has a finite number of infinities X' none of which are odd multiples of $\frac{1}{2}\pi$. In this case the integral is not unconditionally convergent for any value of a.

Thus
$$P\int_0^{\infty} \tan\tfrac{1}{2}x \cos ax \frac{x\,dx}{x^2-\theta^2} = 2\lim_{a=1}\sum_1^{\infty} (-)^{n-1} a^n P\int_0^{\infty} \sin nx \cos ax \frac{x\,dx}{x^2-\theta}$$
$$= \tfrac{1}{2}\pi \lim_{a=1}\sum_1^{\infty} (-)^{n-1} a^n \left\{\cos(n-a)\theta + \cos(n+a)\theta\right\}$$
$$= \pi\cos a\theta \lim_{a=1} \frac{\cos\theta + a}{1+2a\cos\theta+a^2} = \tfrac{1}{2}\pi\cos a\theta.$$

Similarly,
$$P\int_0^{\infty} \cot\tfrac{1}{2}x \cos ax \frac{x\,dx}{x^2-\theta^2} = -\tfrac{1}{2}\pi\cos a\theta.$$

This, again, agrees with § 12, and with the paper in the *Quarterly Journal* referred to above.

Discontinuous Principal Values.

34. We shall now consider some examples in which the conditions of § 25 are not satisfied.

(i.) If $0 < \alpha < a$,

$$P\int_0^\infty \frac{\sin \alpha x}{\cos ax}\,\frac{x\,dx}{1+x^2} = \tfrac{1}{2}\pi\,\frac{\sinh \alpha}{\cosh a}, \tag{1}$$

$$P\int_0 \frac{\cos \alpha x}{\sin ax}\,\frac{x\,dx}{1+x^2} = \tfrac{1}{2}\pi\,\frac{\cosh \alpha}{\sinh a}. \tag{2}$$

These principal values are discontinuous for $\alpha = a$. For, if we put $\alpha = a$ in the first, for instance, we obtain

$$P\int_0^\infty \frac{x \tan ax}{1+x^2}\,dx = \tfrac{1}{2}\pi \tanh a,$$

which is incorrect, the proper value being

$$\frac{\pi}{e^{2a}+1}.$$

Hence (1), which is, after § 23, uniformly convergent in $(0, \alpha' < 1)$ cannot be uniformly convergent in $(0, 1)$.

Now, if ϵ be a small positive quantity,

$$P\int_0^\infty \frac{\sin (a-\epsilon)\,x}{\cos ax}\,\frac{x\,dx}{1+x^2} = P\int_0^\infty \tan ax \cos \epsilon x\,\frac{x\,dx}{1+x^2} - \int_0^\infty \frac{x \sin \epsilon x}{1+x^2}\,dx.$$

The latter integral is, as is well known, discontinuous for $\epsilon = 0$, being $= \frac{1}{2}\pi e^{-\epsilon}$ if $\epsilon > 0$. And it is easy to see that it is not uniformly convergent in an interval including $\epsilon = 0$. For $\dfrac{x}{1+x^2}$ decreases steadily after $x = 1$. Hence, for sufficiently small values of ϵ,

$$\begin{aligned}\int_{2\pi/\epsilon}^\infty \frac{x \sin \epsilon x}{1+x^2}\,dx = \sum_2^\infty \int_{i\pi/\epsilon}^{(i+1)\pi/\epsilon} &> \int_{2\pi/\epsilon}^{4\pi/\epsilon} > \int_{2\pi}^{4\pi} \frac{u \sin u}{\epsilon^2+u^2}\,du \\ &> \int_{2\pi}^{3\pi} \left(\frac{u}{\epsilon^2+u^2} - \frac{u+\pi}{\epsilon^2+(u+\pi)^2}\right) \sin u\,du \\ &> \int_{2\pi}^{3\pi} \frac{\pi\,(u^2+u\pi-\epsilon^2)}{(\epsilon^2+u^2)\left\{\epsilon^2+(u+\pi)^2\right\}} \sin u\,du \\ &> \tfrac{1}{2}\pi^3 \int_{2\pi}^{3\pi} \frac{\sin u}{(u+\pi)^4}\,du,\end{aligned}$$

a positive quantity independent of ϵ. And, however great be A, we can choose ϵ so that $\frac{2\pi}{\epsilon} = A$; and then $\int_A^\infty \frac{x \sin \epsilon x}{1+x^2}\, dx$ is greater than this positive quantity, so that the integral is not uniformly convergent.

On the other hand, $P\int_0^\infty \tan ax \cos \epsilon x \frac{x\, dx}{1+x^2}$ is continuous for $\epsilon = 0$. This does not follow at once from anything which precedes, but is not difficult to prove directly. For, in the first place,

$$P\int_0^\infty \tan ax \cos \epsilon x \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx = P\int_0^\infty \frac{\tan ax \cos \epsilon x}{x\,(1+x^2)}\, dx$$

is uniformly convergent in any finite interval of values of ϵ, and therefore continuous. This follows from the remark at the end of § 23, since $\int^\infty \frac{dx}{x\,(1+x^2)}$ is convergent. Moreover,

$$P\int_0^\infty \tan ax \cos \epsilon x \frac{dx}{x}$$

is continuous for $\epsilon = 0$. For, if $\frac{\epsilon}{a} = \delta$, it is

$$P\int_0^\infty \tan x \cos \delta x \frac{dx}{x} = \sum_0^\infty P\int_{i\pi}^{(i+1)\pi} = \sum_0^\infty P\int_0^\pi \tan x \cos \delta\,(x+i\pi)\, \frac{dx}{x+i\pi}$$

$$= \sum_0^\infty \int_0^{\frac{1}{2}\pi} \tan x \left\{ \frac{\cos \delta\,(x+i\pi)}{x+i\pi} - \frac{\cos \delta\,(i\pi+\pi-x)}{i\pi+\pi-x} \right\} dx.$$

Now the series

$$\sum_1^\infty \left\{ \frac{\cos \delta\,(x+i\pi)}{x+i\pi} - \frac{\cos \delta\,(i\pi+\pi-x)}{i\pi+\pi-x} \right\}$$

is uniformly convergent in $(0, \frac{1}{2}\pi)$ for any small value of $\delta > 0$. Hence we may sum under the sign of integration.

Moreover, it is not difficult to show that

$$\sum_n^\infty \left\{ \frac{\cos \delta\,(x+i\pi)}{x+i\pi} - \frac{\cos \delta\,(i\pi+\pi-x)}{i\pi+\pi-x} \right\}$$

$$= \frac{1}{x+n\pi} + \sum_{n+1}^\infty \left\{ \frac{1}{x+i\pi} + \frac{1}{x-i\pi} \right\} - \int_0^\delta \sin n\pi\delta \frac{\sin\,(\frac{1}{2}\pi-x)\,\delta}{\sin \frac{1}{2}\pi\delta}\, d\delta. \quad (1)$$

The last term is $$\frac{1}{n}\int_0^{n\delta}\sin\pi t\,\frac{\sin\left(\frac{1}{2}\pi-x\right)\frac{t}{n}}{\sin\frac{\pi t}{2n}}\,dt. \qquad (a)$$

Now, if $0<u<\delta$, and δ is sufficiently small,

$$\frac{\sin\left(\frac{1}{2}\pi-x\right)u}{\sin\frac{1}{2}\pi u} \quad (0<x<\tfrac{1}{2}\pi)$$

is positive, and increases steadily as u increases from 0 to δ, and so lies between

$$\frac{\frac{1}{2}\pi-x}{\frac{1}{2}\pi} \text{ and } \frac{\sin\left(\frac{1}{2}\pi-x\right)\delta}{\sin\frac{1}{2}\pi\delta},$$

which differ by a quantity which vanishes with δ. And, if l is the greatest integer contained in $n\delta$, and

$$n\delta=l+\rho,$$

the modulus of (a) is less than

$$\frac{2}{n}\left|\int_{l-1}^{l}\sin\pi t\,\frac{\sin\left(\frac{1}{2}\pi-x\right)\frac{t}{n}}{\sin\frac{1}{2}\pi\frac{t}{n}}\,dt\right|+\frac{1}{n}\left|\int_{l}^{l+\rho}\right|<\frac{C}{n},$$

where C is a quantity independent of n, δ, and x. Hence (1) can be made as small as we please, by choice of n, for all values of δ and x in question.

Hence $\sum_1^\infty$ is uniformly convergent in the domain

$$x=(0,\ \tfrac{1}{2}\pi),\quad \delta=(0,\ \delta_0),$$

where δ_0 is any small positive quantity. It follows that

$$P\int_0^\infty \tan x\cos\delta x\,\frac{dx}{x}=\int_0^{\frac{1}{2}\pi}\tan x\sum_0^\infty\left\{\frac{\cos\delta\,(x+i\pi)}{x+i\pi}-\frac{\cos\delta\,(i\pi+\pi-x)}{i\pi+\pi-x}\right\}dx$$

is a continuous function of δ for $\delta=0$. And, in fact, we find on summing under the integral sign, since

$$\sum_0^\infty\left\{\frac{\cos\delta\,(x+i\pi)}{x+i\pi}-\frac{\cos\delta\,(i\pi+\pi-x)}{i\pi+\pi-x}\right\}=\cot x,$$

that $$P\int_0^\infty \tan x\cos\delta x\,\frac{dx}{x}=\tfrac{1}{2}\pi.$$

This principal value is therefore independent of δ, and changes con-

tinuously, for $\delta = 0$, into

$$P\int_0^\infty \tan x \frac{dx}{x} = \tfrac{1}{2}\pi,$$

as the preceding analysis shows that it should.

Hence
$$P\int_0^\infty \frac{\sin ax}{\cos ax}\,\frac{x\,dx}{1+x^2}$$

has a discontinuity of magnitude $\frac{1}{2}\pi$ to the left of $\alpha = a$.

Cauchy noticed the corresponding discontinuity of

$$P\int_0^\infty \frac{\cos ax}{\sin ax}\,\frac{x\,dx}{1+x^2};$$

which, if $\alpha = a-\epsilon$, is

$$P\int_0^\infty \cot ax \cos \epsilon x \frac{x\,dx}{1+x^2} - \int_0^\infty \frac{x \sin \epsilon x}{1+x^2}\,dx.$$

But his discussion of it cannot be considered satisfactory. For he assumes that the first term is continuous for $\epsilon = 0$. And, moreover, he is content to accept the discontinuity of the second as a fact, without in any way attempting to explain it.

35. (ii.) It is easy to prove that, if

$$\alpha > 1, \quad a > 0, \quad c > 0,$$

$$\int_0^\infty \frac{\alpha \cos ax - \cos (a-c)\,x}{1-2\alpha \cos cx + \alpha^2}\,\frac{dx}{1+x^2} = \tfrac{1}{2}\pi\,\frac{e^{-a}}{\alpha - e^{-c}}, \tag{1}$$

$$P\int_0^\infty \frac{\alpha \cos ax - \cos (a-c)\,x}{1-2\alpha \cos cx + \alpha^2}\,\frac{dx}{1-x^2} = \tfrac{1}{2}\pi\,\frac{\alpha \sin a - \sin (a-c)}{1-2\alpha \cos c + \alpha^2}. \tag{2}$$

We might expect, after our investigations in §§ 27–30, to be able to put $\alpha = 1$ in these formulæ, provided we introduce the sign of the principal value before (1). But this gives

$$P\int_0^\infty \frac{\sin (a-\frac{1}{2}c)\,x}{\sin \frac{1}{2}cx}\,\frac{dx}{1+x^2} = -\pi\,\frac{e^{-a}}{1-e^{-c}},$$

$$P\int_0^\infty \frac{\sin (a-\frac{1}{2}c)\,x}{\sin \frac{1}{2}cx}\,\frac{dx}{1-x^2} = \tfrac{1}{2}\pi\,\frac{\cos (a-\frac{1}{2}c)}{\sin \frac{1}{2}c};$$

both of which are incorrect.

The explanation of this is very simple. For let us consider the simplest case, in which $a = c$. Then

$$\int_0^\infty \frac{\alpha \cos cx - 1}{1-2\alpha \cos cx + \alpha^2}\,\frac{dx}{1+x^2} = \tfrac{1}{2}\pi\,\frac{e^{-c}}{\alpha - e^{-c}} \quad (\alpha > 1),$$

and the limit of this for $\alpha = 1$ is $\dfrac{\pi}{2(e^c-1)}$, whereas its value for $\alpha = 1$ is $-\frac{1}{2}\pi$.

The fact is that
$$\int_a^A \frac{\alpha \cos cx - 1}{1-2\alpha \cos cx + \alpha^2}\,\phi(x)\,dx \tag{i.}$$

is discontinuous for $a = 1$ if (a, A) include any of the points

$$\frac{2n\pi}{c}.$$

And it is easy to see that it is not uniformly convergent. Suppose, *e.g.*, $c = 1$, $a = 0$, $\phi(x) = 1$. Then

$$\int_0^{\xi} \frac{a \cos x - 1}{1 - 2a \cos x + a^2}\, dx = \tan^{-1} \frac{\sin \xi}{a - \cos \xi}.$$

Now $$\tan^{-1} \frac{\sin \xi}{a - \cos \xi} < \sigma$$

involves $$\sin \xi < \tan \sigma \,(a - 1 + \tfrac{1}{2} \sin^2 \xi \ldots);$$

and, however small be ξ, we can choose a value of a so nearly equal to 1 that this inequality is not satisfied.

The integral (i.) is, in fact, substantially Poisson's integral, which is so important in the theory of trigonometrical series.

[36. (iii.) If $\phi(x)$ is a function of x whose derivate $\phi'(x)$ is continuous, the principal value

$$\Phi(a) = P\int_a^A \log\left(1 - \frac{a}{x}\right)^2 \frac{\phi(x)}{x - \beta}\, dx \quad (a < a < A,\ a < \beta < A)$$

is continuous for $a = \beta$. For

$$\Phi(\beta) - \Phi(\beta - \epsilon) = P\int_a^A \log\left(\frac{x - \beta}{x - \beta + \epsilon}\right)^2 \frac{\phi(x)}{x - \beta}\, dx.$$

Now we may replace a, A by $\beta - \rho$, $\beta + \rho$, where ρ is any small fixed positive quantity; for the limits of $\int_a^{\beta-\rho}$, $\int_{\beta+\rho}^A$ for $\epsilon = 0$ are evidently both zero. And

$$\begin{aligned} P\int_{\beta-\rho}^{\beta+\rho} &= P\int_{-\rho}^{\rho} \log\left(\frac{u}{u + \epsilon}\right)^2 \phi(u + \beta)\, \frac{du}{u} \\ &= \phi(\beta)\, P\int_{-\rho}^{\rho} \log\left(\frac{u}{u + \epsilon}\right)^2 \frac{du}{u} + \int_{-\rho}^{\rho} \log\left(\frac{u}{u + \epsilon}\right)^2 \phi'(u_1 + \beta)\, du, \end{aligned}$$

where $-\rho \leq u_1 \leq \rho$. It is easy to see that the last integral tends to zero with ϵ.

But the first is

$$\phi(\beta) \int_0^{\rho} \log\left(\frac{u - \epsilon}{u + \epsilon}\right)^2 \frac{du}{u} = \phi(\beta) \int_0^{\rho/\epsilon} \log\left(\frac{t - 1}{t + 1}\right)^2 \frac{dt}{t};$$

and the limit of this for $\epsilon = 0$ is

$$\phi(\beta) \int_0^{\infty} \log\left(\frac{t - 1}{t + 1}\right)^2 \frac{dt}{t} = -\tfrac{1}{2}\pi^2 \phi(\beta).$$

Hence $$\Phi(\beta - 0) - \Phi(\beta) = \tfrac{1}{2}\pi^2 \phi(\beta);$$

and, similarly, $$\Phi(\beta) - \Phi(\beta + 0) = \tfrac{1}{2}\pi^2 \phi(\beta).$$

We shall frequently meet with discontinuities of this kind when we come to consider the differentiation and integration of principal values.— *November 8th, 1901.*]

Continuity of Principal Values (continued).

37. Theorem 2.—*If $f(x, a)$ be a continuous function of both variables in any finite part of the rectangle*

$$(a, \infty, \beta, \gamma)$$

which does not include any point of any of the curves $x = X_i(a)$, and

$$P\int_a^\infty f(x, a)\,dx$$

be regularly convergent in (β, γ), it will be a continuous function of a in (β, γ).

For, if σ be any assigned positive quantity, we can determine (§ 21) a value of A, a division of (β, γ) into two sets of finite intervals θ, η_i, and a set of positive quantities p_i, such that

$$\left| P\int_A^\infty \right| < \tfrac{1}{3}\sigma$$

in the intervals θ and $\quad \left| P\int_{A-p}^\infty \right| < \tfrac{1}{3}\sigma$

in the intervals η_i.

And, if a_0 be any value of a in (β, γ), we can choose h' so small that a_0 and a_0+h' lie in the same sub-interval. Suppose, for instance, that they lie in η_i. Then

$$P\int_a^{A-p_i}$$

is uniformly convergent in η_i. And the conclusion follows as in § 25.

38. Thus $$P\int_0^\infty \frac{\tan ax}{x}\,dx \tag{1}$$

is regularly convergent in (β, γ) if $0<\beta<\gamma$, and therefore continuous. It is, in fact, $= \frac{1}{2}\pi$ (§ 30). But it is not regularly convergent in $(0, \gamma)$. For $a = 0$ all the curves $x = X_i(a)$ recede to infinity. And it is easy to show, by an argument similar to that used in § 34 in the case of the integral

$$\int_0^\infty \frac{x \sin ax}{1+x^2}\,dx,$$

that, however great be A, we can always determine a positive quantity τ and a value of a such that

$$P\int_A^\infty > \tau,$$

and, moreover,

$$P\int_{A-p}^\infty \Big| > \tau,$$

for all values of p less than any fixed quantity p_0.

It is obvious that (1) is, as a matter of fact, discontinuous for $a = 0$.

On the Exponential Theorem for a Simply Transitive Continuous Group, and the Calculation of the Finite Equations from the Constants of Structure. *By* H. F. Baker. Communicated February 14th, 1901. Received, in revised form, November 28th, 1901.

The present note was originally presented to the London Mathematical Society in February, 1901, in connexion with Mr. Campbell's paper, Vol. xxxiii., p. 285, and had then the purposes of suggesting the methodical use of a certain notation—that of the theory of matrices—and of showing how Mr. Campbell's results follow from Schur's determination of the infinitesimal transformations of a group of given structure (§ 4). Incidentally the theorem (§ 2) here called the exponential theorem was then obtained, and it was stated that it would lead to a method of finding the finite transformations of a group of given structure. The present form of the note differs from the original form by the addition of a verification of this statement, with examples (§§ 3, 5, and the latter part of § 4), and a considerable abbreviation of some parts of the paper whose novelty was stated to consist only in the methods employed.

1. The following notation is employed.

The differential equations satisfied by the functions f in the equations

$$x_i = f_i\,(x^0, a)$$

of a finite continuous group of n variables $x_1, \ldots, x_n$ and r parameters

$a_1, \ldots, a_r$ are written

$$\frac{\partial x}{\partial a} = \xi\psi \quad \text{or} \quad \frac{\partial x}{\partial a}\alpha = \xi,$$

where ξ denotes a matrix of n rows and r columns, whose elements are expressible as functions of $x_1, \ldots, x_n$ only, ψ and α are each matrices of r rows and columns whose elements are functions of $a_1, \ldots, a_r$ only, and they are inverse matrices, so that $\psi\alpha = \alpha\psi = 1$, and $\frac{\partial x}{\partial a}$ denotes a matrix of n rows and r columns whose general element, that in the i-th row and σ-th column, is

$$\frac{\partial x_i}{\partial a_\sigma} \quad \begin{pmatrix} i = 1, \ldots, n \\ \sigma = 1, \ldots, r \end{pmatrix};$$

so that * the equations express that

$$\frac{\partial x_i}{\partial a_\sigma} = \xi_{i1}\psi_{1\sigma} + \ldots + \xi_{ir}\psi_{r\sigma} \quad (i = 1, \ldots, n;\ \sigma = 1, \ldots, r);$$

thus, if f be any function of $x_1, \ldots, x_n$ only, and thence of $a_1, \ldots, a_r$, we have

$$\frac{\partial f}{\partial a_\sigma} = \sum_i \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial a_\sigma} = \sum_{\rho=1}^{r} \psi_{\rho\sigma} P_\rho f,$$

where
$$P_\rho = \xi_{1\rho}\frac{\partial}{\partial x_1} + \ldots + \xi_{n\rho}\frac{\partial}{\partial x_n}$$

is one of the r infinitesimal transformations of the group. It is known that the matrix ξ cannot have its columns linearly connected with coefficients independent of $x_1, \ldots, x_n$, that is, that the n equations

$$\xi(p_1, \ldots, p_n) = 0,$$

namely,
$$\xi_{i1}p_1 + \ldots + \xi_{ir}p_r = 0,$$

cannot all be satisfied with $p_1, \ldots, p_r$ independent of $x_1, \ldots, x_n$; similarly the matrices α and ψ have non-vanishing determinants.

With the infinitesimal transformations P_σ arise also, in expressing the conditions of integrability of the differential equations above, r other infinitesimal transformations, known as those of the parameter group, namely,

$$A_\rho = \alpha_{1\rho}\frac{\partial}{\partial a_1} + \ldots + \alpha_{r\rho}\frac{\partial}{\partial a_r}.$$

Then we have

$$P_\rho P_\sigma - P_\sigma P_\rho = c_{\rho\sigma 1}P_1 + \ldots + c_{\rho\sigma r}P_r, \quad A_\rho A_\sigma - A_\sigma A_\rho = c_{\rho\sigma 1}A_1 + \ldots + c_{\rho\sigma r}A_r,$$

* Thus ξ_{ij} and α_{ij} are the quantities which Lie denotes by ξ_{ji} and α_{ji}.

where the constants $c_{\sigma\rho\tau}$, which we call constants of structure, satisfy the equations

$$\left.\begin{array}{r} c_{\rho\sigma\tau} = -c_{\sigma\rho\tau} \\ \sum\limits_{\tau=1}^{r} (c_{\beta\gamma\tau} c_{\tau\alpha\delta} + c_{\gamma\alpha\tau} c_{\tau\beta\delta} + c_{\alpha\beta\tau} c_{\tau\gamma\delta}) = 0 \end{array}\right\} \quad (\alpha, \beta, \gamma, \delta, \sigma = 1, \ldots, r).$$

In what follows we frequently consider the square matrix of r rows and columns whose (ρ, σ)-th element is

$$E_{\rho\sigma} = \sum_{\tau=1}^{r} c_{\sigma\tau\rho} e_\tau ;$$

denoting this by E, and the matrices obtained from it by replacing $e_1, \ldots, e_r$ respectively by $e_1', \ldots, e_r'$ and $e_1'', \ldots, e_r''$ by E', E'', the relations among the constants of structure are expressed by the equations

$$Ee' + E'e = 0,$$

$$EE'e'' + E'E''e + E''Ee' = 0,$$

holding for all values of the three sets of quantities $(e_1, \ldots, e_r)$, $(e_1', \ldots, e_r')$, $(e_1'', \ldots, e_r'')$. The former also gives $Ee = 0$, and shows that the determinant of the matrix E vanishes identically. The infinitesimal transformations of a group are determined by the "matrix ξ" of the group; the group in the variables $e_1, \ldots, e_r$, of r parameters, whose "matrix ξ" is the (negative of the) matrix E is that called the *adjoint* group. In place of the parameters $a_1, \ldots, a_r$, we may use any r independent functions of these; in particular, we may use so-called *canonical* parameters $e_1, \ldots, e_r$, which are such that the increment of a function f of $x_1, \ldots, x_n$, when the parameters receive small increments $e_1\delta t, \ldots, e_r\delta t$ respectively proportional to their then values, shall be given by

$$\delta f = \delta t\,(e_1 P_1 f + \ldots + e_r P_r f) ;$$

supposing then, in the formulæ above, e_σ written for a_σ, and comparing with

$$\delta f = \sum_\sigma \sum_\rho \psi_{\rho\sigma} P_\rho f \delta e_\sigma ,$$

we infer, since the infinitesimal transformations P_ρ are not linearly connected with coefficients independent of $x_1, \ldots, x_n$, that

$$\sum_\sigma \psi_{\rho\sigma} \delta e_\sigma = \delta t e_\rho ,$$

or, since $\qquad \delta e_\sigma = e_\sigma \delta t,$

that $\qquad \psi e = e,$

which gives also $\qquad \alpha e = e.$

In other words, when the parameters $e_1, \ldots, e_r$ are canonical,

$$(\psi-1)\,e=0 \quad \text{and} \quad (a-1)\,e=0,$$

and each of the determinants $|\psi-1|$, $|a-1|$ is zero.

When in the equations of a group

$$x_i=f_i\,(x^0,\, a)$$

we change to new variables ξ and ξ^0, eliminating x and x^0 by means of equations

$$\xi_i=\phi_i\,(x), \quad \xi_i^0=\phi_i\,(x^0),$$

and so obtaining the group expressed, say, by

$$\xi_i=F_i\,(\xi^0,\, a),$$

it is generally said that the group has been *transformed* by

$$\xi_i=\phi_i\,(x).$$

This use of the frequent word transformation appears inconvenient, and we venture to suggest that the group be said to be *translated.* Considering now a simply transitive group in which the number of variables is equal to the number r of parameters, it will not generally be the case that the matrix ξ satisfies the equations

$$\xi x=x,$$

namely, the r equations $\xi_{\sigma 1}x_1+\ldots+\xi_{\sigma r}x_r=x_\sigma$;

but the group can be translated into such a group. For, if

$$x_i'=f_i\,(x,\, k)$$

be the finite equations of the group, with canonical parameters $k_1, \ldots, k_r$, and, instead of $x_1, \ldots, x_r$, we take variables $e_1, \ldots, e_r$ determined from

$$x_i=f_i\,(x^0,\, e),$$

and, correspondingly, instead of $x_1', \ldots, x_r'$ we take variables $e_1', \ldots, e_r'$ determined from

$$x_i'=f_i\,(x^0,\, e'),$$

where, in both, $x_1^0, \ldots, x_r^0$ are the same arbitrarily assigned quantities, then, as

$$f_i\,(x^0,\, e')=x_i'=f_i\,(x,\, k)=f_i\left[f(x^0,\, e),\, k\right]=f_i\left[x^0,\, \phi\,(e,\, k)\right],$$

we have

$$e_\sigma'=\phi_\sigma\,(e,\, k),$$

which are the equations for the group in the new variables. In fact, the group is translated into its first parameter group.

In dealing with a simply transitive group of r variables and parameters, we shall therefore denote its infinitesimal transformations by

$$A_\sigma = a_{1\sigma}\frac{\partial}{\partial e_1}+\ldots+a_{r\sigma}\frac{\partial}{\partial e_r},$$

the variables being denoted by $(e_1, \ldots, e_r)$; and shall suppose $ae = e$. We shall call these canonical variables. Their use will be found to effect considerable simplifications.

We now give a sketch of Schur's determination of the infinitesimal transformations of a simply transitive group of given structure, the variables being canonical. From the equations

$$(A_\rho A_\sigma) = \sum_{\tau=1}^{r} c_{\rho\sigma\tau} A_\tau,$$

where A_σ has the form just put down, we immediately deduce

$$\frac{\partial\psi_{\nu h}}{\partial e_k}-\frac{\partial\psi_{\nu k}}{\partial e_h} = \sum_\rho\sum_\sigma c_{\rho\sigma\nu}\psi_{\rho h}\psi_{\sigma k},$$

where, as before, $\psi = a^{-1}$;

from these also, if $D = \sum_{k=1}^{r} e_k \frac{\partial}{\partial e_k}$,

we at once find $D\psi+\psi-1 = E\psi$,

where by the differential coefficient of the matrix ψ is meant the matrix of the differential coefficients of its elements.

If we now assume that $\psi_{\rho\sigma}$ is capable of expansion, for sufficiently small values of $e_1, \ldots, e_r$, in the form

$$\psi_{\rho\sigma} = A_{\rho\sigma}+B_{\rho\sigma}+C_{\rho\sigma}+\ldots,$$

where $A_{\rho\sigma}$ is a homogeneous polynomial in $e_1, \ldots, e_r$ of zero dimension, that is, a constant, $B_{\rho\sigma}$ a homogeneous polynomial of dimension unity, $C_{\rho\sigma}$ of dimension 2, and so on, so that we may write

$$\psi = A+B+C+\ldots,$$

where each of the letters $A, B, C, \ldots$ denotes a matrix, then, on substitution in the equation

$$E\psi = D\psi+\psi-1,$$

we obtain

$$EA+EB+EC+\ldots = B+2C+3D+\ldots+A-1+B+C+\ldots;$$

and therefore, equating terms of the same dimension,

$$A-1=0, \quad EA=2B, \quad EB=3C, \quad \dots;$$

so that

$$\psi = 1+\frac{E}{2!}+\frac{E^2}{3!}+\dots .$$

Conversely, starting with the r^3 constants of structure $c_{\rho\sigma\tau}$, satisfying the necessary relations, and defining a matrix E as before, so that

$$E_{\rho\sigma} = \sum_{\tau} c_{\sigma\tau\rho} e_{\tau},$$

and hence

$$Ee = 0,$$

and putting ψ equal to the matrix expressed by the series in E just given, so that

$$\psi e = e,$$

and the determinant of ψ, reducing to 1, for

$$e_1 = 0 = \dots = e_r,$$

is not zero, we immediately verify that

$$E\psi + 1 - \psi = D\psi.$$

From this we can deduce, putting

$$a = \psi^{-1},$$

and defining the infinitesimal transformations A_{σ} as before, that

$$(A_{\rho}A_{\sigma}) = \sum_{\tau} c_{\rho\sigma\tau} A_{\tau},$$

which shows that these infinitesimal transformations determine a simply transitive group of the given structure. (*Cf.* Lie, *Transformationsgruppen*, III., § 144.)

A similar use may be made of the matrix notation to determine the simply transitive group, known to exist (Lie, *Transformationsgruppen*, I., p. 380), which is reciprocal to the given one. In fact, the equations

$$\sum_{\mu=1}^{r}\left(a_{\mu\sigma}\frac{\partial \gamma_{\nu\rho}}{\partial e_{\mu}} - \gamma_{\mu\rho}\frac{\partial a_{\nu\sigma}}{\partial e_{\mu}}\right) = 0 \quad (\rho, \sigma = 1, \dots, r),$$

which express that the transformations

$$\Gamma_{\rho} = \gamma_{1\rho}\frac{\partial}{\partial e_1}+\dots+\gamma_{r\rho}\frac{\partial}{\partial e_r}$$

of the reciprocal group are commutable with those of the original group, give very easily the equation

$$D\chi + \chi = \chi\psi^{-1},$$

where χ is the inverse of the matrix γ and D is the homogeneous operator before used. Putting here

$$\chi = A + A_1 + A_2 + \ldots,$$

where A_λ is a matrix of r rows and columns of which each element is a homogeneous polynomial in $e_1, \ldots, e_r$ of degree λ, and, equating terms of like dimension, we find

$$\chi = A\left(1 - \frac{E}{2!} + \frac{E^2}{3!} - \frac{E^3}{4!} + \ldots\right),$$

where A is a matrix of constants; the deduction is precisely like that of the solution of the differential equation

$$\left(x\frac{dy}{dx} + y\right)\frac{e^x - 1}{x} = y.$$

Introducing the condition that χ shall reduce to unity when

$$e_1 = 0 = \ldots = e_r,$$

and denoting, for a few lines, by ψ' the same function of $(-e_1, -e_2, \ldots, -e_r)$ that ψ is of $(e_1, \ldots, e_r)$, we have then $\chi = \psi'$; which is Lie's result, *Transformationsgruppen*, III., p. 651. Putting, as we shall frequently do in what follows,

$$\Delta = 1 + E + \frac{E^2}{2!} + \frac{E^3}{3!} + \ldots,$$

we immediately verify that $\qquad \alpha' = \alpha\Delta,$

where $\qquad \alpha' = \psi'^{-1}$

is the same function of $(-e_1, \ldots, -e_r)$ that α is of $(e_1, \ldots, e_r)$. The verification is like that of the identity $(e^x - 1)/x = e^x(e^{-x} - 1)/(-x)$.

2. From this we proceed now to deduce the exponential theorem so called in the superscription of this paper, namely, we proceed to show that, if

$$e''_\sigma = \phi_\sigma(e, e')$$

be the finite equations of the group under consideration, with canonical variables e and e'' and canonical parameters e', and if Δ', Δ'' denote the same functions of e', e'' respectively that Δ is of e, then

$$\Delta'' = \Delta'\Delta.$$

The equation $\qquad \alpha' = \alpha\Delta$

gives $\qquad \alpha'_{\mu\rho} = \sum_{\nu=1}^{r} \Delta_{\nu\rho}\alpha_{\mu\nu},$

and hence, if
$$A_\sigma = \sum_{\tau=1}^{r} a'_{\tau\sigma} \frac{\partial}{\partial e_\tau},$$
gives
$$A'_\sigma = \sum_\tau \Delta_{\tau\sigma} A_\tau.$$

Now the transformations A'_σ are commutable with A_σ; thus we have
$$\begin{aligned} 0 = (A'_\rho A_\sigma) = A'_\rho A_\sigma - A_\sigma A'_\rho &= \sum_\tau \Delta_{\tau\rho} A_\tau A_\sigma - \sum_\tau \Delta_{\tau\rho} A_\sigma A_\tau - \sum_\tau (A_\sigma \Delta_{\tau\rho}) A_\tau \\ &= \sum_\tau \Delta_{\tau\rho} \sum_\lambda c_{\tau\sigma\lambda} A_\lambda - \sum_\lambda (A_\sigma \Delta_{\lambda\rho}) A_\lambda \\ &= -\sum_\lambda A_\lambda \left[A_\sigma \Delta_{\lambda\rho} - \sum_\tau c_{\tau\sigma\lambda} \Delta_{\tau\rho}\right]; \end{aligned}$$
and therefore, as the transformations A_λ are linearly independent, we have
$$A_\sigma \Delta_{\lambda\rho} = \sum_\tau c_{\tau\sigma\lambda} \Delta_{\tau\rho}.$$

Hence, if
$$P = e'_1 A_1 + \ldots + e'_r A_r$$
be the general infinitesimal transformation of the group (A_σ) with canonical parameters $(e'_1, \ldots, e'_r)$, we have
$$P\Delta_{\lambda\rho} = \sum_\tau E'_{\lambda\tau} \Delta_{\tau\rho} = (E'\Delta)_{\lambda\rho},$$
where
$$E'_{\lambda\tau} = \sum_\sigma c_{\tau\sigma\lambda} e'_\sigma.$$
This is the same as
$$P\Delta = E'\Delta,$$
the quantities $e'_1, \ldots, e'_r$ being quite arbitrary, and gives
$$P^2\Delta = E'P\Delta = E'^2\Delta,$$
and, in general,
$$P^k\Delta = E'^k\Delta.$$

Thus, if
$$e''_\sigma = \phi_\sigma(e, e')$$
be the finite transformations of the group (A_σ) with canonical parameters $(e'_1, \ldots, e'_r)$, and the value of Δ obtained by replacing $e_1, \ldots, e_r$ respectively by $e''_1, \ldots, e''_r$ be denoted by Δ'' or by $\Delta_{\phi(e,e')}$, we have
$$\begin{aligned} \Delta'' = \Delta_{\phi(e,e')} &= \Delta + P\Delta + \frac{1}{2!}P^2\Delta + \ldots \\ &= \Delta + E'\Delta + \frac{1}{2!}E'^2\Delta + \ldots \\ &= \Delta'\Delta, \end{aligned}$$
where Δ' is obtained from Δ by replacing $e_1, \ldots, e_r$ respectively by $e'_1, \ldots, e'_r$.

This is the theorem referred to.

Two simple applications suggest themselves—

(i.) $\Delta_{\phi[\phi(e,e'),e_1]} = \Delta_{e_1}\Delta_{\phi(e,e')} = \Delta_{e_1}\Delta_{e'}\Delta_e = \Delta_{\phi(e',e_1)}\Delta_e = \Delta_{\phi[e,\phi(e',e_1)]}$,

in accordance with the known result

$$\phi\left[\phi(e, e'), e_1\right] = \phi\left[e, \phi(e', e_1)\right];$$

(ii.) $\Delta_{-\phi(-e',-e)} = \left[\Delta_{\phi(-e',-e)}\right]^{-1} = \left[\Delta_{-e}\Delta_{-e'}\right]^{-1} = \Delta_{-e'}^{-1}\Delta_{-e}^{-1}$

$$= \Delta_{e'}\Delta_e = \Delta_{\phi(e,e')},$$

in accordance with the known result

$$\phi(-e', -e) = -\phi(e, e').$$

How far these equations are really proved, that is, how far the equation

$$\Delta_h = \Delta_k$$

enables us to infer $\qquad h_\sigma = k_\sigma$,

is now to be considered.

3. Starting with given constants of structure, satisfying the necessary conditions, we can, by direct processes, calculate the matrices Δ_e, $\Delta_{e'}$; then the equation

$$\Delta_{e''} = \Delta_{e'}\Delta_e$$

furnishes r^2 equations for e_1'', ..., e_r''; if these were thence obtained, say in the forms

$$e_\sigma'' = \phi_\sigma(e, e'),$$

the finite equations of the group would be known. From Schur's theory there exist infinitesimal transformations, and therefore finite equations corresponding to given constants of structure. Thus these equations necessarily have solutions. But the question is (i.) whether the r^2 equations are equivalent to as many as r independent equations, since otherwise other conditions than

$$\Delta_{e''} = \Delta_{e'}\Delta_e$$

must be necessary to define the forms of e_1'', ..., e_r''; (ii.) whether, if they are equivalent to as many as r independent equations, these equations have unique or only ambiguous solutions.

We know beforehand the answer to at least the first of these equations, for Lie has proved that, given the infinitesimal transformations, for the finite equations of a simply transitive group of r parameters and $r-m$ special (*ausgezeichnete*) infinitesimal transformations,

m independent analytical equations can be put down without integration, and that these can be supplemented by $r-m$ quadratures (*Transformationsgruppen*, III., p. 634, and I., p. 376). And a comparison of his proof (p. 628) with the equation

$$a' = a\Delta,$$

used by us, shows that, in fact, of the r^2 equations arising for $e_1'', \ldots, e_r''$, from

$$\Delta_{e''} = \Delta_{e'}\Delta_e,$$

just m are independent (*cf. Transformationsgruppen*, I., p. 509).

In what follows the result obtained arises naturally in a different form, which would appear to be of more utility for the actual purpose of calculating $e_1'', \ldots, e_r''$.

First, if the columns of the matrix E are not linearly connected with coefficients independent of the variables $e_1, \ldots, e_r$, in which case the adjoint group has r linearly independent infinitesimal transformations, and the original group has no (special) transformations commutable with the others, it is shown that the equations

$$\Delta_{e''} = \Delta_{e'}\Delta_e$$

have only one set of solutions for $e_1'', \ldots, e_r''$ within the range for which the group is simply transitive.

Next, if the columns of E are linearly connected with constant coefficients, and expressible by m of themselves, in which case the adjoint group is of m parameters and the original group has $r-m$ special infinitesimal transformations, it is shown that linear functions $f_1, \ldots, f_r$ of the original canonical variables $e_1, \ldots, e_r$ can be taken so that the infinitesimal transformations of the group reduce to

$$B_p = A_p' + u_{m+1,p}\frac{\partial}{\partial f_{m+1}} + \ldots + u_{r,p}\frac{\partial}{\partial f_r} \quad (p = 1, \ldots, m),$$

$$B_P = \frac{\partial}{\partial f_P} \qquad (P = m+1, \ldots, r),$$

where
$$A_p' = a_{1p}'\frac{\partial}{\partial f_1} + \ldots + a_{mp}'\frac{\partial}{\partial f_m};$$

here all the mr coefficients $a_{1p}', \ldots, a_{mp}', u_{m+1,p}, \ldots, u_{r,p}$ are functions only of $f_1, \ldots, f_m$, and directly calculable from the constants of structure, and the m transformations A_p' generate a group of m parameters in the variables $f_1, \ldots, f_m$, which therefore, as follows from this form of the infinitesimal transformations, is also simply transitive. The differential equations for the finite equations of the group

are now

$$\frac{df_q}{\sum_{p=1}^{m} a'_{qp} f'_p} = \ldots = \frac{df_Q}{\sum_{p=1}^{m} u_{Qp} f'_p + f'_Q} = dt \quad \begin{pmatrix} q = 1, \ldots, m \\ Q = m+1, \ldots, r \end{pmatrix},$$

which are to be integrated, so that $f_e = f_e^{(0)}$ when $t = 0$, and then $f_e^{(0)}$, f_e, and t are to be replaced by f_e, f''_e, and 1. Thus it is obvious that, if $f_1, \ldots, f_m$ are found as functions of t, from the first m fractions or otherwise, and the results substituted in the coefficients $u_{Q,p}$, the values of $f_{m+1}, \ldots, f_r$ are at once obtainable by $r-m$ independent quadratures. We consider then how to find $f_1, \ldots, f_m$. If the constants of structure of the group (A'_p) be denoted by $d_{q\lambda p}$, and, similar to the formation of E for the group (A_e), we form a matrix M of m rows and columns for the group (A'_p), wherein

$$M_{pq} = \sum_{\lambda=1}^{m} d_{q\lambda p} f_\lambda,$$

and put $$\Delta^{(1)} = 1 + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \ldots,$$

the equation $$\Delta_{e''} = \Delta_{e'} \Delta_e$$

is at once found to lead to $$\Delta^{(1)}_{f''} = \Delta^{(1)}_{f'} \Delta^{(1)}_{f}.$$

If then the group (A'_p) has no special infinitesimal transformations, it follows from the previous case that this equation determines $f''_1, \ldots, f''_m$ without ambiguity; and then, as above, the finite equations of the original group are determined.

In this case the equations arising from

$$\Delta_{e''} = \Delta_{e'} \Delta_e,$$

other than $$\Delta^{(1)}_{f''} = \Delta^{(1)}_{f'} \Delta^{(1)}_{f},$$

do not contain the variables $f''_{m+1}, \ldots, f''_r$ as will be seen. Thus the $r-m$ integrations are really necessary and the equation

$$\Delta_{e''} = \Delta_{e'} \Delta_e$$

does not suffice alone to determine $e''_1, \ldots, e''_r$.

If, however, the group (A'_p) have $m-m_1$ special infinitesimal transformations, we can introduce variables $f^{(1)}_1, \ldots, f^{(1)}_m$, linear functions of $f_1, \ldots, f_m$, to reduce the infinitesimal transformations of the

group (A'_p) to the forms

$$B'_{p_1} = A''_{p_1} + u^{(1)}_{m_1+1,\,p_1}\frac{\partial}{\partial f^{(1)}_{m_1+1}} + \ldots + u^{(1)}_{m,\,p_1}\frac{\partial}{\partial f^{(1)}_{m}} \quad (p_1 = 1, \ldots, m_1),$$

$$B'_{P_1} = \frac{\partial}{\partial f^{(1)}_{P_1}} \qquad (P_1 = m_1+1, \ldots, m),$$

wherein
$$A''_{p_1} = a''_{1p_1}\frac{\partial}{\partial f^{(1)}_1} + \ldots + a''_{m_1 p_1}\frac{\partial}{\partial f^{(1)}_{m_1}},$$

the coefficients $a''_{1p_1}, \ldots, a''_{m_1,p_1}, u^{(1)}_{m_1+1,p_1}, \ldots, u^{(1)}_{m,p_1}$ being functions only of $f^{(1)}_1, \ldots, f^{(1)}_{m_1}$, and the infinitesimal transformations (A''_{p_1}) determining a simply transitive group of m_1 parameters. When the finite transformations of this group are found, it follows as before that the remaining transformations of the group (A'_p) are determinable very conveniently by $m-m_1$ quadratures.

In accordance with Lie's theorem, these $m-m_1$ quadratures should not be actually necessary to finding m independent equations for $f''_1, \ldots, f''_m$. And, in fact, beside the equation

$$\Delta^{(1)}_{f''} = \Delta^{(1)}_{f}\,\Delta^{(1)}_{f'},$$

leading to the equation

$$\Delta^{(2)}_{11} = \Delta^{(2)}_{1}\,\Delta^{(2)},$$

containing only the variables $f^{(1)}_1, \ldots, f^{(1)}_{m_1}$, there arise from $\Delta_{e''} = \Delta_{e'}\Delta_{e}$ other equations containing the variables $f^{(1)}_{m_1+1}, \ldots, f^{(1)}_{m}$. However, our argument shows that the equations

$$\Delta^{(2)}_{11} = \Delta^{(2)}_{1}\,\Delta^{(2)}$$

have unique solutions; and the carrying out of the $m-m_1$ quadratures appears to be more convenient than the solution of analytical equations, which, as will be seen, are generally not algebraical.

The next stage is therefore the determination of the finite equations of the group (A''_{p_1}). In this, as before, distinction arises according as the group has special infinitesimal transformations or not. And so on continually, the whole number of quadratures employed being $(r-m)+(m-m_1)+(m_1-m_2)+\ldots$, and the equations used beside the quadratures having unique solutions.

[*January* 30*th*, 1902.—When the group has special infinitesimal transformations, the rule here suggested for calculating the finite equations requires the calculation of the coefficients $u_{m+1,p}, \ldots, u_{r,p}$ in the infinitesimal transformations (p. 108, below), and, apart from its simplicity, may be regarded as adding to Lie's theorem (*Trg.*, III.,

p. 634) only the discovery that the elements of the matrix Δ are equivalent with his function-system ϕ_{k_r}. But when there are no special transformations the rule proceeds straight from the constants of structure to all the finite equations.

Further, to show that the case considered in the paper in which the group (A'_p) has special transformations can arise, the following example may be given. It can be verified that there is a set of constants of structure, leading to a matrix E, satisfying $E^3 = -e_1 E^2$, given by

$$E = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ e_2 & -e_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e_3 & 0 & -e_1 & 0 \end{Bmatrix},$$

wherein the remaining variable e_4 does not enter. For this $r = 4$, $m = 3$, $m_1 = 2$. By the rule here given we find for the finite equations $e''_1 = e_1 + e'_1$, $e''_3 = e_3 + e'_3$, with

$$e''_2 \{1-\lambda(-e_1-e'_1)\}$$
$$= (e_1+e'_1)\left[\frac{e_2}{e_1}\{1-\lambda(-e_1)\}\lambda(-e'_1)+\frac{e'_2}{e'_1}\{1-\lambda(-e'_1)\}\right].$$

where $\lambda(x)$ is the exponential function $\exp(x)$. By these and p. 108 (below) we find for the infinitesimal transformations

$$\frac{\partial}{\partial e_1}+u\frac{\partial}{\partial e_2}-\tfrac{1}{2}e_3\frac{\partial}{\partial e_4},\quad v\frac{\partial}{\partial e_2},\quad \frac{\partial}{\partial e_3}+\tfrac{1}{2}e_1\frac{\partial}{\partial e_4},\quad \frac{\partial}{\partial e_4},$$

where $\qquad u = e_2\{1-e_1-\lambda(-e_1)\}/e_1\{1-\lambda(-e_1)\}$

and $\qquad v = e_1/\{1-\lambda(-e_1)\}.$

By a quadrature we then find for the remaining finite equation

$$e''_4 = \tfrac{1}{2}(e_1e'_3-e_3e'_1)+e_4+e'_4.]$$

4. We have now to justify the reductions explained in § 3.

Consider the r^2 linear equations for $x_1, \ldots, x_r$ expressed by

$$\sum_{\tau=1}^{r} c_{\tau\sigma\rho}x_\tau = 0 \quad (\rho, \sigma = 1, 2, \ldots, r):$$

the left side is equal to $\qquad -\sum_{\tau=1}^{r} c_{\sigma\tau\rho}x_\tau,$

and may be expressed by $\qquad -(E_x)_{\rho\sigma},$

being the (ρ, σ)-th element of the matrix E_x obtained from E by replacing $e_1, \ldots, e_r$ by $x_1, \ldots, x_r$.

Let $r-m$ be the number of linearly independent sets of common solutions of these equations, and let

$$a_{1P}, \ldots, a_{rP} \quad (P = m+1, \ldots, r)$$

denote any one of these sets, one determinant of order $r-m$ from the matrix of these quantities being other than zero.

The columns of the matrix E may be connected by linear equations with coefficients independent of $e_1, \ldots, e_r$; to one such equation would correspond then r equations

$$A_1 E_{\rho 1} + \ldots + A_r E_{\rho r} = 0,$$

for

$$\rho = 1, \ldots, r,$$

the coefficients $A_1, \ldots, A_r$ being independent of $e_1, \ldots, e_r$, and the same for all values of ρ. Suppose that the greatest number of columns which are not connected in this way is μ, and that all other columns are linearly expressible by such μ columns, with coefficients independent of $e_1, \ldots, e_r$; so that there are $r-\mu$ sets of equations

$$A_{Q,1} E_{\rho 1} + \ldots + A_{Q,r} E_{\rho r} = 0 \quad (Q = \mu+1, \ldots, r),$$

wherein the coefficients $A_{Q1}, \ldots, A_{Qr}$ are independent of $e_1, \ldots, e_r$, and the same for all values of ρ, and one determinant of order $r-\mu$ from the matrix of these coefficients is other than zero.

It is, then, at once obvious that

$$m = \mu.$$

For we have $Ee' = -E'e$, namely, for every value of $\rho, = 1 \ldots r$,

$$E_{\rho 1} e_1' + \ldots + E_{\rho r} e_r' = -E_{\rho 1}' e_1 - \ldots - E_{\rho r}' e_r;$$

thus any linear equation with coefficients $e_1', \ldots, e_r'$ independent of $e_1, \ldots, e_r$ connecting the columns of E leads to a set, $e_1', \ldots, e_r'$, of solutions of the r^2 equations $E_{\rho\sigma}' = 0$, and conversely; or $r-m = r-\mu$. The equality of these numbers with that of the special transformations follows from the identity

$$\left(P_\rho, \sum_{\sigma=1}^{r} e_\sigma' P_\sigma\right) = \sum_{\tau=1}^{r} E_{\tau\rho}' P_\tau.$$

Now form a matrix a of r rows and columns of non-vanishing determinant of which the last $r-m$ columns consist of the quantities $a_{1P}, \ldots, a_{rP}$—since one of the determinants of order $r-m$ from these columns is other than zero, this is possible in an infinite number of ways—and take

$$F = a^{-1}Ea, \quad (f_1, \ldots, f_r) = a^{-1}(e_1, \ldots, e_r), \cdots$$

or, say
$$f = a^{-1}e,$$
so that
$$\begin{aligned} F_{\rho\sigma} &= \sum_{\lambda} (a^{-1})_{\rho\lambda} (Ea)_{\lambda\sigma} = \sum_{\lambda}\sum_{\mu} (a^{-1})_{\rho\lambda} E_{\lambda\mu} a_{\mu\sigma} = \sum_{\lambda}\sum_{\mu}\sum_{\tau} (a^{-1})_{\rho\lambda} c_{\mu\tau\lambda} e_{\tau} a_{\mu\sigma} \\ &= \sum_{\lambda}\sum_{\mu}\sum_{\tau}\sum_{h} (a^{-1})_{\rho\lambda} c_{\mu\tau\lambda} a_{\tau h} f_h a_{\mu\sigma} = \sum_{h} d_{\sigma h \rho} f_h, \end{aligned}$$
if only
$$d_{\sigma h \rho} = \sum_{\lambda} (a^{-1})_{\rho\lambda} \sum_{\mu}\sum_{\tau} c_{\mu\tau\lambda} a_{\tau h} a_{\mu\sigma} ;$$

and the matrix F is precisely the same function of the r^3 quantities $d_{\sigma h \rho}$ and the r quantities f_h as is E respectively of the r^3 quantities $c_{\sigma h \rho}$ and the r quantities e_h.

The quantities $d_{\sigma h \rho}$ are, in fact, constants of structure of a certain system of infinitesimal transformations; for, if, with the usual notation
$$A_{\sigma} = a_{1\sigma} \frac{\partial}{\partial e_1} + \ldots + a_{r\sigma} \frac{\partial}{\partial e_r},$$
we take
$$B_{\rho} = a_{1\rho} A_1 + \ldots + a_{r\rho} A_r,$$
we have
$$\begin{aligned} (B_{\rho} B_{\sigma}) &= \sum_{\lambda, \mu} a_{\lambda\rho} a_{\mu\sigma} (A_{\lambda} A_{\mu}) = \sum_{\lambda, \mu}\sum_{\tau} a_{\lambda\rho} a_{\mu\sigma} c_{\lambda\mu\tau} A_{\tau} \\ &= \sum_{h}\sum_{\lambda, \mu}\sum_{\tau} a_{\lambda\rho} a_{\mu\sigma} c_{\lambda\mu\tau} (\bar{a})^{-1}_{\tau h} B_h = \sum_{h} \Big[\sum_{\tau} (a^{-1})_{h\tau} \sum_{\lambda, \mu} a_{\lambda\rho} a_{\mu\sigma} c_{\lambda\mu\tau} \Big] B_h \\ &= \sum_{h} d_{\rho\sigma h} B_h ; \end{aligned}$$

and it is of interest to express these infinitesimal transformations in terms of $f_1, \ldots, f_r$; by the definition
$$B_{\sigma} = \sum_{\tau} (aa)_{\tau\sigma} \frac{\partial}{\partial e_{\tau}},$$
and hence, putting
$$\gamma = aa,$$
and remarking that
$$f = a^{-1}e$$
gives
$$\frac{\partial}{\partial e_{\tau}} = (a^{-1})_{1\tau} \frac{\partial}{\partial f_1} + \ldots + (a^{-1})_{r\tau} \frac{\partial}{\partial f_r},$$
we have
$$B_{\sigma} = \sum_{\tau} \gamma_{\tau\sigma} \frac{\partial}{\partial e_{\tau}} = \sum_{\lambda} (a^{-1}\gamma)_{\lambda\sigma} \frac{\partial}{\partial f_{\lambda}} = \sum_{\lambda} \beta_{\lambda\sigma} \frac{\partial}{\partial f_{\lambda}}, \text{ say,}$$
where
$$\beta = a^{-1}\gamma = a^{-1}aa$$
is, in virtue of
$$ae = e,$$
such that
$$\beta f = a^{-1}aaf = a^{-1}ae = a^{-1}e = f;$$
in other words, $e_1, \ldots, e_r$ being canonical variables, so are $f_1, \ldots, f_r$, and

the constants of structure $d_{\sigma h\rho}$ belong to the infinitesimal transformations

$$B_\sigma = \sum_\lambda \beta_{\lambda\sigma} \frac{\partial}{\partial f_\lambda},$$

of which the "matrix ξ" is the matrix

$$\beta = a^{-1} a a.$$

That the constants $d_{\sigma h\rho}$ are constants of structure follows also of course more directly; taking $(e_1', \ldots, e_r')$ and $(e_1'', \ldots, e_r'')$ arbitrarily, and putting

$$f = a^{-1} e, \quad f' = a^{-1} e', \quad f'' = a^{-1} e'',$$

and correspondingly $F' = a^{-1} E' a, \quad F'' = a^{-1} E'' a,$

we have $$aFf' = Ee';$$

so that from $$Ee' + E'e = 0$$

follows $$Ff' + F'f = 0,$$

equivalent to $$d_{\sigma h\rho} = - d_{h\sigma\rho};$$

while $$FF'f'' = a^{-1} E a F' f'' = a^{-1} E E' a f'' = a^{-1} E E' e'';$$

so that $$FF'f'' + F'F''f + F''Ff' = 0,$$

equivalent to $$\sum_\tau (d_{\beta\gamma\tau} d_{\tau\alpha\delta} + d_{\gamma\alpha\tau} d_{\tau\beta\delta} + d_{\alpha\beta\tau} d_{\tau\gamma\delta}) = 0.$$

In this transformation from the matrix E to the matrix F we have used no property of the constant matrix a, save that its determinant is not zero; taking account of the special definition of the last $r-m$ columns of this matrix, we have

$$d_{P\sigma\rho} = \sum_\lambda (a^{-1})_{\rho\lambda} \sum_\mu \sum_\tau c_{\mu\tau\lambda} a_{\tau\sigma} a_{\mu P} = 0 \quad (P = m+1, \ldots, r),$$

$$d_{pP\rho} = \sum_\lambda (a^{-1})_{\rho\lambda} \sum_\mu \sum_\tau c_{\mu\tau\lambda} a_{\tau P} a_{\mu p} = - \sum_\lambda (a^{-1})_{\rho\lambda} \sum_\mu \sum_\tau c_{\mu\tau\lambda} a_{\tau p} a_{\mu P} = 0$$
$$(p = 1, \ldots, m);$$

so that the matrix F has the form

$$F = \begin{pmatrix} M & 0 \\ N & 0 \end{pmatrix},$$

wherein the last m columns are identically zero because of

$$d_{P\sigma\rho} = 0,$$

and in M, N the quantities $f_{m+1}, \ldots, f_r$ do not occur because of

$$d_{pP\rho} = 0;$$

in fact, $$M_{pq} = \sum_{\lambda=1}^{m} d_{q\lambda p} f_\lambda, \quad N_{Pq} = \sum_{\lambda=1}^{m} d_{q\lambda P} f_\lambda$$
$$(p, q = 1, \dots, m;\ P = m+1, \dots, r),$$

M being a matrix of m rows and columns, and N a matrix of $r-m$ rows and m columns.

Consider now more particularly the forms of the infinitesimal transformations $B_1, \dots, B_r$ in the variables $f_1, \dots, f_r$. Schur's expansion
$$a = 1 - \frac{E}{2} + \frac{E^2}{12} + \dots$$
leads, because $$\beta = a^{-1}\alpha a, \quad F = a^{-1}Ea,$$
and therefore $$F^k = a^{-1}E^k a,$$
to the expansion $$\beta = 1 - \frac{F}{2} + \frac{F^2}{12} + \dots,$$
which shows that the elements of the last $r-m$ columns of the matrix β are all zero except the elements β_{PP} for
$$P = m+1, \dots, r,$$
which are unities; so that $B_{m+1}, \dots, B_r$ reduce respectively to $\frac{\partial}{\partial f_{m+1}}, \dots, \frac{\partial}{\partial f_r}$; and, in fact, since
$$F^k = \begin{pmatrix} M^k & 0 \\ NM^{k-1} & 0 \end{pmatrix},$$
the matrix $$\beta^{-1} = 1 + \frac{F}{2!} + \frac{F^2}{3!} + \dots$$
has the form $$\beta^{-1} = \begin{pmatrix} \psi' & 0 \\ \phi' & 1 \end{pmatrix},$$
where $$\psi' = 1 + \frac{M}{2!} + \frac{M^2}{3!} + \dots;$$
thus $$a' = \psi'^{-1}$$
is the "matrix ξ" of a group in the m variables $f_1, \dots, f_m$ with constants of structure d_{qlp} for
$$p, q, l = 1, 2, \dots, m;$$
also $$\phi' = N\left(\frac{1}{2!} + \frac{M}{3!} + \frac{M^2}{4!} + \dots\right);$$
thus, if we put $$u = -\phi'\psi'^{-1},$$

we have
$$\beta = \begin{pmatrix} \psi' & 0 \\ -u\psi' & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \alpha' & 0 \\ u & 1 \end{pmatrix};$$

and therefore
$$\left.\begin{aligned} B_p &= A'_p + u_{m+1,\,p}\frac{\partial}{\partial f_{m+1}} + \ldots + u_{r,\,p}\frac{\partial}{\partial f} \\ B_P &= \frac{\partial}{\partial f_P} \end{aligned}\right\}$$
$$(p = 1, \ldots, m;\ P = m+1, \ldots, r),$$

where
$$A'_p = \alpha'_{1p}\frac{\partial}{\partial f_1} + \ldots + \alpha'_{mp}\frac{\partial}{\partial f_m}.$$

The quantities $\alpha'_{q,\,p}$, $u_{P,\,p}$, being formed from the matrix F, are functions of $f_1, \ldots, f_m$ only; if we put
$$D' = f_1\frac{\partial}{\partial f_1} + \ldots + f_m\frac{\partial}{\partial f_m},$$
we immediately verify from the series given above for ϕ' and ψ' that
$$D'\phi' + \phi' = N\psi',$$
which conversely determines ϕ' when the series for ψ' is given; putting
$$f_1 = \lambda_1 t, \ \ldots, \ f_m = \lambda_m t,$$
and regarding ϕ', ψ' and N as fractions of t only, this equation leads to
$$t\frac{d\phi'}{dt} + \phi' = N\psi';$$
so that
$$\phi' = \frac{1}{t}\int_0^t N\psi' dt,$$
which means
$$(\phi')_{R,\,p} = \frac{1}{t}\int_0^t (N\psi')_{R,\,p}\, dt \quad \begin{pmatrix} R = 1, 2, \ldots, r-m \\ p = 1, 2, \ldots, m \end{pmatrix};$$
hence, putting
$$\lambda_1 = f_1/t, \ \ldots, \ \lambda_m = f_m/t,$$
the form of ϕ', and thus of $u = -\phi'\alpha'$, as functions of $f_1, \ldots, f_m$ is determined; this includes the solution of the equations
$$\frac{\partial\phi'_{R,\,p}}{\partial f_q} - \frac{\partial\phi'_{R,\,q}}{\partial f_p} = \sum_{i=1}^{m}\sum_{k=1}^{m} c_{ikR}\,(\beta^{-1})_{ip}\,(\beta^{-1})_{kq},$$
satisfied by the quantities $\phi'_{R,\,p}$, $\phi'_{R,\,q}$ (*cf.* Campbell, *Proc. Lond. Math. Soc.*, Vol. XXXIII., pp. 290, 292. The group (A'_p) is the parameter group of the adjoint group.)

Consider now the square matrix M in the matrix

$$F = \begin{pmatrix} M & 0 \\ N & 0 \end{pmatrix};$$

it is a function of the constants of structure d_{pql} for $p, q, l = 1, \ldots, m$, and the variables $f_1, \ldots, f_m$, similar in form to E as a function of $c_{\rho\sigma\tau}$ for $\rho, \sigma, \tau = 1, \ldots, r$ and $e_1, \ldots, e_r$. If the columns of the matrix M be subject to linear equations with coefficients independent of $f_1, \ldots, f_m$, let m_1 be the number of these columns independent in this respect, the other columns of M being such linear functions of these. Then, as we have shown, we can take a matrix a_1 of m rows and columns, and non-vanishing determinant, and make a linear transformation

$$f^{(1)} = a_1^{-1} f,$$

affecting only the variables $f_1, \ldots, f_m$, such that the matrix $a_1^{-1} M a_1$ shall take the form

$$a_1^{-1} M a_1 = \begin{pmatrix} M_1 & 0 \\ N_1 & 0 \end{pmatrix},$$

wherein M_1, N_1 are linear functions of the new variables $f^{(1)}$, containing, however, only $f_1^{(1)}, \ldots, f_{m_1}^{(1)}$, the coefficients being new constants of structure $d^{(1)}_{p_1 q_1 p}$, in which

$$p = 1, \ldots, m, \quad p_1, q_1 = 1, \ldots, m_1,$$

M_1 consisting of m_1 rows and columns, and N_1 of $m - m_1$ rows and m_1 columns. The infinitesimal transformations A'_p for

$$p = 1, \ldots, m$$

will then naturally be replaced by linear functions of themselves taking the forms

$$\left.\begin{aligned} B'_{p_1} &= A''_{p_1} + u^{(1)}_{m_1+1, p_1} \frac{\partial}{\partial f^{(1)}_{m_1+1}} + \ldots + u^{(1)}_{m, p_1} \frac{\partial}{\partial f^{(1)}_m} \\ B'_{P_1} &= \frac{\partial}{\partial f^{(1)}_{P_1}} \end{aligned}\right\}$$

$$(p_1 = 1, \ldots, m_1;\ P_1 = m_1 + 1, \ldots, m),$$

where
$$A''_{p_1} = a''_{1 p_1} \frac{\partial}{\partial f_1^{(1)}} + \ldots + a''_{m_1 p_1} \frac{\partial}{\partial f^{(1)}_{m_1}},$$

and $a''_{1, p_1}, \ldots, a''_{m_1 p_1}$; $u^{(1)}_{m_1+1, p_1}, \ldots, u^{(1)}_{m, p_1}$ are functions of $f_1^{(1)}, \ldots, f^{(1)}_{m_1}$ only.

Take now
$$b = \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix},$$
so that
$$b^{-1} = \begin{pmatrix} a_1^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$
and form $F_1 = b^{-1}Fb = \begin{pmatrix} a_1^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} M & 0 \\ N & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1^{-1}Ma_1 & 0 \\ Na_1 & 0 \end{pmatrix}$;

then, as
$$c = ab = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{a}a_1 & \mathrm{b} \\ \mathrm{c}a_1 & \mathrm{d} \end{pmatrix}$$
has the same last $r-m$ columns as the matrix a, and a non-vanishing determinant equal to the product of the determinants of a and a_1, and
$$F_1 = c^{-1}Ec,$$
it is clear that we can argue with the matrix F_1 as we argued with F; the infinitesimal transformations of the group will have the form
$$A''_{p_1} + u^{(1)}_{m_1+1,p_1}\frac{\partial}{\partial f^{(1)}_{m_1+1}} + \dots + u^{(1)}_{m,p_1}\frac{\partial}{\partial f^{(1)}_{m}} + u_{m+1,p_1}\frac{\partial}{\partial f_{m+1}} + \dots + u_{r,p_1}\frac{\partial}{\partial f_r},$$
$$\frac{\partial}{\partial f^{(1)}_{P_1}} + u_{m+1,P_1}\frac{\partial}{\partial f_{m+1}} + \dots + u_{r,P_1}\frac{\partial}{\partial f_r},$$
$$\frac{\partial}{\partial f_P}.$$

where A''_{p_1} involves differentiations in regard to $f_1^{(1)}, \dots, f_{m_1}^{(1)}$, the coefficients being functions of these variables only, the suffix p_1 has the values $1, 2, \dots, m_1$, the suffix P_1 the values $m_1+1, \dots, m$, and the suffix P the values $m+1, \dots, r$; and $u^{(1)}_{m_1+1,p_1}, \dots, u^{(1)}_{m,p_1}$ are functions only of $f_1^{(1)}, \dots, f_{m_1}^{(1)}$, while $u_{m+1,p_1}, \dots, u_{r,p_1}, u_{m+1,P_1}, \dots, u_{r,P_1}$ are functions only of $f_1^{(1)}, \dots, f_m^{(1)}$. Also the matrix F_1 will have the form
$$F_1 = \left\{\begin{matrix} \begin{pmatrix} M_1 & 0 \\ N_1 & 0 \end{pmatrix} & 0 \\ Na_1 & 0 \end{matrix}\right\}.$$

If now the columns of the matrix M_1 can be expressed linearly by m_2 of them, with coefficients independent of $f_1^{(1)}, \dots, f_{m_1}^{(1)}$, we can take a matrix a_2 of m_1 rows and columns, such that
$$a_2^{-1}M_1a_2 = \begin{pmatrix} M_2 & 0 \\ N_2 & 0 \end{pmatrix},$$

and then with
$$b_1 = \left\{ \begin{matrix} \begin{pmatrix} a_2 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 1 \end{matrix} \right\}$$

we can form
$$F_2 = b_1^{-1} F_1 b_1,$$

which is of the form
$$\left\{ \begin{matrix} \begin{pmatrix} a_2^{-1} M_1 a_2 & 0 \\ N_1 a_2 & 0 \end{pmatrix} & 0 \\ w & 0 \end{matrix} \right\},$$

and associated with this there will be a certain linear transformation of the variables $f_1^{(1)}, \ldots, f_{m_1}^{(1)}$ to variables $f_1^{(2)}, \ldots, f_{m_1}^{(2)}$.

This process can be continued indefinitely.

Further, an equation $\quad \Delta_{e''} = \Delta_{e'} \Delta_e$

leads to
$$a^{-1}\Delta_{e''} a = (a^{-1}\Delta_{e'} a)(a^{-1}\Delta_e a),$$

and $a^{-1}\Delta a = 1 + \dfrac{a^{-1}Ea}{1!} + \dfrac{a^{-1}E^2 a}{2!} + \ldots = 1 + F + \dfrac{F^2}{2!} + \ldots = \begin{pmatrix} \Delta^{(1)} & 0 \\ H^{(1)} & 1 \end{pmatrix}$,

where
$$\Delta^{(1)} = 1 + \frac{M}{1!} + \frac{M^2}{2!} + \frac{M^3}{3!} + \ldots,$$
$$H^{(1)} = N\left(1 + \frac{M}{2!} + \frac{M^2}{3!} + \ldots\right),$$

while
$$\begin{pmatrix} \Delta_{11}^{(1)} & 0 \\ H_{11}^{(1)} & 1 \end{pmatrix} = \begin{pmatrix} \Delta_1^{(1)} & 0 \\ H_1^{(1)} & 1 \end{pmatrix} \begin{pmatrix} \Delta^{(1)} & 0 \\ H^{(1)} & 1 \end{pmatrix} = \begin{pmatrix} \Delta_1^{(1)} \Delta^{(1)} & 0 \\ H^{(1)}\Delta^{(1)} + H^{(1)} & 1 \end{pmatrix},$$

and so the equation $\quad \Delta_{e''} = \Delta_{e'} \Delta_e$

gives
$$\Delta_{11}^{(1)} = \Delta_1^{(1)} \Delta^{(1)};$$

but the equation
$$H_{11}^{(1)} = h_1^{(1)} \Delta^{(1)} + H^{(1)}$$

does not involve the variables $f''_{m+1}, \ldots, f''_r$.

Also, from
$$F_1 = \left\{ \begin{matrix} \begin{pmatrix} M_1 & 0 \\ N_1 & 0 \end{pmatrix} & 0 \\ Na_1 & 0 \end{matrix} \right\}, \quad F_1^k = \left\{ \begin{matrix} \begin{pmatrix} M_1^k & 0 \\ N_1 M_1^{k-1} & 0 \end{pmatrix} & 0 \\ Na_1 \begin{pmatrix} M_1 & 0 \\ N_1 & 0 \end{pmatrix}^{k-1} & 0 \end{matrix} \right\},$$

we have
$$c^{-1}\Delta c = 1 + \frac{F_1}{1!} + \frac{F_1^2}{2!} + \ldots = \left\{ \begin{matrix} \begin{pmatrix} \Delta^{(2)} & 0 \\ H^{(2)} & 1 \end{pmatrix} & 0 \\ K^{(2)} & 1 \end{matrix} \right\},$$

where $\quad \Delta^{(2)} = 1 + \frac{M_1}{1!} + \frac{M_1^2}{2!} + \dots, \quad H^{(2)} = N_1\left(1 + \frac{M_1}{2!} + \dots\right),$

$$K^{(2)} = Na_1\left\{1 + \begin{pmatrix} M_1 & 0 \\ N_1 & 0 \end{pmatrix}\frac{1}{2!} + \begin{pmatrix} M_1 & 0 \\ N_1 & 0 \end{pmatrix}^2\frac{1}{3!} + \dots\right\}.$$

The equation $\quad \Delta_{e''} = \Delta_{e'}\Delta_e$

gives
$$\left\{\begin{matrix} \begin{pmatrix} \Delta_{11}^{(2)} & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ K_{11}^{(2)} & 1 \\ & \dots \end{matrix}\right\} = \left\{\begin{matrix} \begin{pmatrix} \Delta_1^{(2)}\Delta^{(2)} & 0 \\ H_1^{(2)}\Delta^{(2)} + H^{(2)} & 1 \end{pmatrix} & 0 \\ K_1^{(2)}\begin{pmatrix} \Delta^{(2)} & 0 \\ H^{(2)} & 1 \end{pmatrix} + K^{(2)} & 1 \\ & \dots \end{matrix}\right\};$$

and therefore, beside the equation

$$\Delta_{11}^{(2)} = \Delta_1^{(2)}\Delta^{(2)}$$

used above, also the equation

$$K_{11}^{(2)} = K_1^{(2)}\begin{pmatrix} \Delta^{(2)} & 0 \\ H^{(2)} & 1 \end{pmatrix} + K^{(2)},$$

which involves the variables $f^{(1)}_{m_1+1}, \dots, f^{(1)}_m$, beside $f^{(1)}_1, \dots, f^{(1)}_{m_1}$. A proof that this equation led to unique values for $f^{(1)}_{m_1+1}, \dots, f^{(1)}_m$ would dispose of the $m - m_1$ quadratures used above.

We have now to prove that when the columns of the matrix E are not linearly connected with coefficients independent of $e_1, \dots, e_r$, the equation
$$\Delta_{e''} = \Delta_{e'}\Delta_e$$
has unique solutions for $e_1'', \dots, e_r''$.

Let, if possible, $(h_1, \dots, h_r)$ and $(k_1, \dots, k_r)$ be two sets of quantities such that
$$\Delta_h = \Delta_k.$$

Then we have $\quad \Delta_k^{-1}\Delta_h = 1, \quad$ or $\quad \Delta_{-k}\Delta_h = 1,$

or
$$\Delta_{\phi(h,\,-k)} = 1,$$

as follows from § 2; or, if $\; l_\sigma = \varphi_\sigma(h, -k),$

we have $\quad \Delta_l = 1;$

this, however, is equivalent to

$$\psi_l E_l = \left(1 + \frac{E_l}{2!} + \frac{E_l^2}{3!} + \dots\right)E_l = 0;$$

the determinant of the matrix ψ_l, reducing to unity when

$$l_1 = 0 = \dots = l_r,$$

is not identically zero; therefore, multiplying by ψ_i^{-1}, we obtain

$$E_i = 0.$$

We have, however, shown that these r^2 equations have no solution other than

$$l_1 = 0 = \ldots = l_r,$$

provided, as here assumed, the columns of E_i be not linearly connected with constant coefficients.

Now the quantities $$l_\sigma = \phi_\sigma(h, -k)$$

are those which occur in the group equations

$$x'_\sigma = f_\sigma(x^0, l)$$

as the result of eliminating the variables x from the two sets of equations

$$x_\sigma = f_\sigma(x^0, h), \quad x'_\sigma = f_\sigma(x, -k);$$

Thus we have shown that the equation

$$\Delta_h = \Delta_k$$

requires that these two sets of equations lead to

$$x'_\sigma = f_\sigma(x^0, 0).$$

Within the range for the parameters within which the group is simply transitive, these last equations

$$x'_\sigma = f_\sigma(x^0, 0)$$

require $$x'_\sigma = x^0_\sigma.$$

But the equations $$x'_\sigma = f_\sigma(x, -k)$$

are the same as $$x_\sigma = f_\sigma(x', k),$$

the parameters k being canonical, and therefore give

$$x_\sigma = f_\sigma(x^0, k),$$

so that $$f_\sigma(x^0, k) = f_\sigma(x_0, h).$$

These equations, however, give within the range for which the group is simply transitive simply

$$h_\sigma = k_\sigma,$$

which is therefore shown to follow from

$$\Delta_h = \Delta_k.$$

5. We now give some examples of the actual determination of the finite equations. (See also the Example at the end of § 3.)

In general we are to reduce Δ, by means of the equation satisfied by the matrix E, to a polynomial of finite order in E. The coefficients in this polynomial will be functions of the unknown quantities $e_1, \dots, e_r$ entering in E. For instance, if

$$E^\mu = \lambda+\lambda_1 E+\dots+\lambda_{\mu-1}E^{\mu-1},$$

and the roots $\theta_1, \dots, \theta_\mu$ of the equation

$$\theta^\mu = \lambda+\lambda_1\theta+\dots+\lambda_{\mu-1}\theta^{\mu-1}$$

be all different, we have

$$\Delta \begin{vmatrix} 1 & 1 & \dots & 1 \\ \theta_1 & \theta_2 & \dots & \theta_\mu \\ \dots & \dots & \dots & \dots \\ \theta_1^{\mu-1} & \theta_2^{\mu-1} & \dots & \theta_\mu^{\mu-1} \end{vmatrix}$$

$$= \begin{vmatrix} e^{\theta_1} & \dots & e^{\theta_\mu} \\ \theta_1 & \dots & \theta_\mu \\ \dots & \dots & \dots \\ \theta_1^{\mu-1} & \dots & \theta_\mu^{\mu-1} \end{vmatrix} + \begin{vmatrix} 1 & \dots & 1 \\ e^{\theta_1} & \dots & e^{\theta_\mu} \\ \theta_1^2 & \dots & \theta_\mu^2 \\ \dots & \dots & \dots \\ \theta_1^{\mu-1} & \dots & \theta_\mu^{\mu-1} \end{vmatrix} E+\dots+ \begin{vmatrix} 1 & \dots & 1 \\ \theta_1 & \dots & \theta_\mu \\ \dots & \dots & \dots \\ \theta_1^{\mu-2} & \dots & \theta_\mu^{\mu-2} \\ e^{\theta_1} & \dots & e^{\theta_\mu} \end{vmatrix} E^{\mu-1}.$$

Example 1.—When the constants of structure are all zero, the equation

$$\Delta_{e''} = \Delta_{e'}\Delta_e$$

reduces to $1 = 1$; in this case variables $f_1, \dots, f_r$ can be chosen to reduce the finite equations of the group to

$$f_s'' = f_s + f_s'.$$

Example 2.—The group generated by the infinitesimal transformations

$$P_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad P_2 = \frac{\partial}{\partial y}, \quad P_3 = \frac{\partial}{\partial z},$$

for which $\quad (P_1P_2) = P_2, \quad (P_1P_3) = 0, \quad (P_2P_3) = 0,$

has P_3 for special transformation, and its matrix E is

$$E = \begin{pmatrix} 0 & 0 & 0 \\ e_2 & -e_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

requiring no transformation of variables; it satisfies

$$E^2 = \begin{pmatrix} 0 & 0 & 0 \\ -e_1 e_2 & e_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -e_1 E;$$

so that

$$\Delta = 1 + E - \frac{e_1 E}{2!} + \frac{e_1^2 E}{3!} - \frac{e_1^3 E}{4!} + \ldots = 1 + \frac{1-\exp(-e_1)}{e_1} E,$$

$$= 1 + AE, \text{ say,}$$

where
$$A = \frac{1-\exp(-e_1)}{e_1},$$

or
$$\Delta = \begin{pmatrix} 1 & 0 & 0 \\ Ae_2 & 1 - Ae_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the finite equations of the group we therefore have

$$\begin{pmatrix} 1 & 0 & 0 \\ A''e_2'' & 1-A''e_1'' & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ A'e_2' & 1-A'e_1' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ Ae_2 & 1-Ae_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ A'e_2' + (1-A'e_1')\, Ae_2 & (1-A'e_1')(1-Ae_1) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Comparing these, we obtain

$$A''e_2'' = A'e_2' + (1-A'e_1')\, Ae_2,$$

$$A''e_1'' = A'e_1' + Ae_1 - AA'e_1'e_1,$$

giving $\exp(-e_1'') = (1-A'e_1')(1-Ae_1) = \exp(-e_1')\exp(-e_1)$,

or, since $e_1'' = e_1$, when $e_1' = 0$,

$$e_1'' = e_1' + e_1,$$

and
$$e_2''/e_1'' = \frac{A'e_2' + (1-A'e_1')\, Ae_2}{A'e_1' + Ae_1 - Ae_1 A'e_1'},$$

while for the variable e_3 we have, in accordance with the general theory,

$$e_3'' = e_3' + e_3.$$

For the transformations

$$P_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad P_2 = \frac{\partial}{\partial y}, \quad P_3 = \frac{\partial}{\partial z},$$

the " matrix ξ " is $\qquad \xi = \begin{pmatrix} 1 & 0 & 0 \\ -y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

and gives $\qquad \xi\,(x,\ y,\ z) = (x,\ -xy+y,\ z)\,;$

so that the variables are not canonical. It can, however, be easily verified that the above are the correct forms of the finite equations in canonical variables (*cf.* p. 94).

Example 3.—Taking the case of three infinitesimal transformations satisfying the equations

$$(P_1P_2) = P_1, \quad (P_1P_3) = 2P_2, \quad (P_2P_3) = P_3,$$

so that $\qquad c_{121} = 1, \quad c_{122} = 0, \quad c_{123} = 0, \quad c_{131} = 0, \quad c_{132} = 2,$

$$c_{133} = 0, \quad c_{231} = 0, \quad c_{232} = 0, \quad c_{233} = 1,$$

we have $\qquad E = \begin{pmatrix} e_2 & -e_1 & 0 \\ 2e_3 & 0 & -2e_1 \\ 0 & e_3 & -e_2 \end{pmatrix},$

which satisfies $\qquad E^3 = \mu E,$

where $\qquad \mu = e_2^2 - 4e_1e_3,$

and thus

$$\Delta = 1 + E\left(1 + \frac{\mu}{3!} + \frac{\mu^2}{5!} + \frac{\mu^3}{7!} + \dots\right) + E^2\left(\frac{1}{2!} + \frac{\mu}{4!} + \frac{\mu^2}{6!} + \dots\right)$$

$$= 1 + AE + BE^2, \text{ say,}$$

where $\qquad A = \dfrac{\sinh(\sqrt{\mu})}{\sqrt{\mu}}, \quad B = \dfrac{\cosh(\sqrt{\mu}) - 1}{\mu}.$

Squaring the matrix E and substituting in this form for Δ, we find

$$\Delta_e = \begin{pmatrix} 1 + Ae_2 + B(e_2^2 - 2e_1e_3), & -Ae_1 - Be_2e_1, & 2Be_1^2 \\ 2Ae_3 + 2Be_2e_3, & 1 - 4Be_1e_3, & -2Ae_1 + 2Be_1e_2 \\ 2Be_3^2, & Ae_3 - Be_2e_3, & 1 - Ae_2 + B(e_2^2 - 2e_1e_3) \end{pmatrix}.$$

We are now to multiply together $\Delta_{e'}$ and Δ_e, and solve for e_1'', e_2'', e_3'' from the equation

$$\Delta_{e''} = \Delta_{e'}\,\Delta_e.$$

As the formation of the product $\Delta_{e'}\Delta_e$ is perfectly easy, it will be sufficient, writing e_1, e_2, e_3 for e_1'', e_2'', e_3'', to consider the solution for

e_1, e_2, e_3 of an equation

$$\Delta_e = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix},$$

wherein the elements σ_{ij} are known. This gives

$$\sigma_{21}+2\sigma_{32} = 4Ae_3, \quad 2\sigma_{12}+\sigma_{23} = -4Ae_1, \quad \sigma_{11}-\sigma_{33} = 2Ae_2,$$

$$\sigma_{21}-2\sigma_{32} = 4Be_2e_3, \quad \sigma_{23}-2\sigma_{12} = 4Be_1e_2, \quad 1-\sigma_{22} = 4Be_1e_3,$$

$$\sigma_{13} = 2Be_1^2, \quad \sigma_{31} = 2Be_3^2, \quad \sigma_{11}+\sigma_{33}-2 = 2B(e_2^2-2e_1e_3);$$

and therefore $$\frac{e_1}{-\sigma_{12}-\frac{1}{2}\sigma_{23}} = \frac{e_2}{\sigma_{11}-\sigma_{33}} = \frac{e_3}{\frac{1}{2}\sigma_{21}+\sigma_{32}} = \frac{1}{2A},$$

and thus, if the (known) denominators of the first three fractions be denoted by M_1, M_2, M_3 respectively, we have

$$\frac{2\sigma_{11}+2\sigma_{33}-4}{M_2^2-2M_1M_3} = \frac{2\sigma_{13}}{M_1^2} = \frac{2\sigma_{31}}{M_3^2} = \frac{\sigma_{21}-2\sigma_{32}}{M_2M_3} = \frac{\sigma_{23}-2\sigma_{12}}{M_1M_2} = \frac{1-\sigma_{22}}{M_1M_3} = \frac{B}{A^2},$$

which involves five necessary relations among the given quantities $\sigma_{11}, \sigma_{12}, \ldots$, but determines for us only the ratio B/A^2; denoting its value by M, so that M is given, we have, recalling the values of A and B,

$$\frac{\cosh(\sqrt{\mu})-1}{\sinh^2(\sqrt{\mu})} = M,$$

or $$\cosh(\sqrt{\mu}) = \frac{1}{M}-1,$$

and hence $$\mu = \left(\log\frac{1-M+\sqrt{1-2M}}{M}\right)^2;$$

as a function of M this is single valued save near $M=0$, in the neighbourhood of which all its branches increase indefinitely; thus, if we limit ourselves to finite values of e_1, e_2, e_3, and therefore finite values of

$$\mu = e_2^2-4e_1e_3,$$

and recall the fact that the values of e_1, e_2, e_3 are assigned for a certain value of M (in the original notation e_1'', e_2'', e_3'' are to reduce to e_1, e_2, e_3 for $e_1'=0$, $e_2'=0$, $e_3'=0$), we have an unique determination for μ, and therefore for A, which is a series in μ; and therefore also for

$$e_1 = M_1/2A, \quad e_2 = M_2/2A, \quad e_3 = M_3/2A.$$

Thus the equation $$\Delta_{e''} = \Delta_{e'}\Delta_e$$

is proved to be sufficient to determine e_1'', e_2'', e_3'' uniquely in terms of e_1, e_2, e_3 and e_1', e_2', e_3'.

We have alluded to the fact that a group called simply transitive may be so only for a limited range of the parameters. As bearing on the little considered relations of Lie's theory to the theory of functions, it is worth while to point out that precisely the same ambiguity which would arise in the value of μ above if infinite values of e_1, e_2, e_3 were not excluded, would also arise, save for a similar limitation, in the translation of the group to canonical variables. Take, for instance,

$$P_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad P_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

$$P_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z},$$

which give $(P_1P_2) = P_1$, $(P_1P_3) = 2P_2$, $(P_2P_3) = P_3$.

The finite equations are then, with canonical parameters,

$$x = \frac{x_0(c+e_2 s)+2e_1 s}{-2e_3 s x_0 + c - e_2 s}, = F(x_0, e), \text{ say}, \quad y = F(y_0, e), \quad z = F(z_0, e),$$

where $\quad s = \dfrac{1}{\sqrt{\mu}}\sinh \tfrac{1}{2}\sqrt{\mu}, \quad c = \cosh \tfrac{1}{2}\sqrt{\mu}, \quad \mu = e_2^2 - 4e_1 e_3.$

In attempting to translate to the first parameter group in canonical variables as explained above (§ 1), namely, to the equations

$$f_1 = \phi_1(e, e'), \quad f_2 = \phi_2(e, e'), \quad f_3 = \phi_3(e, e'),$$

where e_1, e_2, e_3 are old variables, f_1, f_2, f_3 the new variables, and e_1', e_2', e_3' the parameters, we find, putting

$$H = f_2^2 - 4f_1 f_3, \quad \sigma = \frac{\sin \tfrac{1}{2}\sqrt{H}}{\sqrt{H}}, \quad \gamma = \cosh \tfrac{1}{2}\sqrt{H},$$

that

$$\frac{f_1\sigma}{e_1 sc' + e_1' s'c + (e_1 e_2' - e_2 e_1')\, ss'} = \frac{f_2\sigma}{e_2 sc' + e_2' s'c + 2ss'\,(e_1 e_3' - e_3 e_1')}$$

$$= \frac{f_3\sigma}{e_3 sc' + e_3' s'c + ss'\,(e_2 e_3' - e_3 e_2')}$$

$$= \frac{\gamma}{cc' + ss'\,(e_2 e_2' - 2e_1' e_3 - 2e_1 e_3')}.$$

Denoting this by

$$\sigma f_1/A_1 = \sigma f_2/A_2 = \sigma f_3/A_3 = \gamma/A, = \lambda, \text{ say},$$

we have $$H = \frac{\lambda^2}{\sigma^2}(A_2^2 - 4A_1A_3),$$

or $$\sinh^2 \tfrac{1}{2}\sqrt{H} = \lambda^2 (A_2^2 - 4A_1A_3), \quad \cosh^2 \tfrac{1}{2}\sqrt{H} = \lambda^2 A^2,$$

and therefore, as $$A^2 - (A_2^2 - 4A_1A_3) = 1,$$

it follows that $$\lambda^2 = 1.$$

When therefore H is determined from
$$\cosh \sqrt{H} = 2A^2 - 1,$$

which gives $$H = \left[\log (2A^2 - 1 + 2A\sqrt{A^2 - 1})\right]^2,$$

we have $$\sigma = \sinh \tfrac{1}{2}\sqrt{H}/H, \quad \gamma = \cosh \tfrac{1}{2}\sqrt{H},$$

and $$f_1 = \frac{A_1}{A}\,\frac{\gamma}{\sigma}, \quad f_2 = \frac{A_2}{A}\,\frac{\gamma}{\sigma}, \quad f_3 = \frac{A_3}{A}\,\frac{\gamma}{\sigma}.$$

The ambiguity of H is clearly the same as that before arising for μ.

6. In these concluding sections (6, 7) we give other applications of the matrix notation not directly connected with the exponential theorem.

First, if $$\Gamma_i f = \gamma_{1i}\frac{\partial f}{\partial e_1} + \ldots + \gamma_{ri}\frac{\partial f}{\partial e_r} = 0 \quad (i = 1, \ldots, m)$$

be a complete system of homogeneous linear partial differential equations of the first order which is invariant under the substitutions of the simply transitive group (A_e), we show that there corresponds to this system a sub-group of m parameters of the reciprocal group (A'_e) whose component infinitesimal transformations are linear functions, generally not with constant coefficients, of $\Gamma_1 f, \ldots, \Gamma_r f$ (*cf.* Lie, *Transformationsgruppen*, I., pp. 387, 436), finding the infinitesimal transformations of this sub-group.

Secondly, as a particular case, we find a relation connecting the infinitesimal transformations of invariant sub-groups respectively of the original and the reciprocal group, and show that by use of canonical variables the infinitesimal transformations of an invariant sub-group take a certain simple form.

Thirdly, we sketch Darboux's investigation of the conditions of a homogeneous contact transformation, showing that the algebra is precisely the same as that arising in the investigation of the two forms, known as Riemann's and Weierstrass's forms, of the relations holding for the periods of an Abelian function.

For the first, the equations

$$(\Gamma_i A_\sigma) = \lambda_{i\sigma 1}\Gamma_1 + \ldots + \lambda_{i\sigma m}\Gamma_m \quad \begin{pmatrix} i = 1, \ldots, m \\ \sigma = 1, \ldots, r \end{pmatrix},$$

which express the invariance of the equations

$$\Gamma_1 f = 0, \ \ldots, \ \Gamma_m f = 0,$$

under the group (A_σ), lead easily, if

$$L_{ji} = \lambda_{i1j}e_1 + \ldots + \lambda_{irj}e_r$$

be the element of a matrix denoted by L, to the equation

$$D\gamma + a\gamma - \gamma = -\gamma L,$$

where D is the homogeneous operator before used; and this, using the equation we proved (§ 1),

$$D\psi' + \psi' = \psi' a,$$

where ψ' relates to the reciprocal simply transitive group, gives

$$D(\psi'\gamma) + \psi'\gamma L = 0.$$

Now, as L vanishes when $e_1 = 0 = \ldots = e_r$, assuming a form

$$L = L_1 + L_2 + \ldots,$$

where L_k is a matrix whose elements are homogeneous polynomials in $e_1, \ldots, e_r$ of dimension k, it is easily proved that a matrix P of m rows and columns, of the form

$$P = 1 + P_1 + P_2 + \ldots,$$

where P_k is homogeneous in $e_1, \ldots, e_r$ of dimension k, can be found such that

$$DP = LP.$$

Then $D(\psi'\gamma P) = D(\psi'\gamma)P + \psi'\gamma DP = \left[D(\psi'\gamma) + \psi'\gamma L\right]P = 0$;

so that

$$\psi'\gamma P = H,$$

where H is a matrix of constants, being the value of the matrix γ, when

$$e_1 = 0 = \ldots = e_r.$$

Thus

$$\gamma P = a'H,$$

or

$$\sum_{i=1}^{m} P_{ij}\Gamma_i f = \sum_{\tau=1}^{r} H_\tau A'_\tau f,$$

proving the result stated, the infinitesimal transformations on the right constituting, as we easily see, a sub-group of the reciprocal group (A'_σ).

For the second, if $(\Gamma_1, \ldots, \Gamma_m)$ be an invariant sub-group of the group (A_s), the quantities $\lambda_{i\sigma 1}, \ldots, \lambda_{i\sigma m}$ are constants, and the matrix L is homogeneously of the first degree. Thus the equation

$$DP = LP$$

leads to $$P = 1 + L + \frac{L^2}{2!} + \frac{L^3}{3!} + \ldots.$$

Further, if the constants H_{ri} be such that

$$\Gamma_i = \sum_r H_{ri} A_r,$$

then $$\gamma = aH,$$

and H is the same matrix as before, being the value of γ when

$$e_1 = 0 = \ldots = e_r.$$

Thus the previous equation $\gamma P = \alpha' H$

becomes $$aHP = \alpha' H.$$

Here the columns of the matrix $\alpha' H$ give respectively the coefficients in the infinitesimal transformations of an invariant sub-group of the reciprocal group (A'_s), those of aH in a corresponding invariant sub-group of the group (A_s). In particular, the equation

$$a\Delta = a',$$

previously used to deduce the exponential theorem, is a case, namely, when $L = E$ and $H = 1$. In general the constants H_{ri} must satisfy certain conditions expressing that (Γ_i) is an invariant sub-group.

Changing the notation slightly, to agreement with the earlier part of this paper, let $$B_\rho = a_{1\rho} A_1 + \ldots + a_{r\rho} A_r \quad (\rho = 1, \ldots, r)$$

be infinitesimal transformations of the group (A_s), the determinant of the matrix of the constants $a_{s\rho}$ being other than zero, and

$$(B_\rho B_\sigma) = \sum_\tau d_{\rho\sigma\tau} B_\tau.$$

Change the variables, as before, to the quantities

$$f = a^{-1} e,$$

so that $$F = a^{-1} E a$$

is the matrix for which $$F_{\rho\sigma} = \sum_\tau d_{\sigma\tau\rho} f_\tau.$$

Further, suppose that $(B_1, \ldots, B_m)$ generate an invariant sub-group,

so that every constant $d_{\rho p P}$ for which $P > m$, $p < m+1$ is zero for

$$\rho = 1, \ldots, r;$$

thus, in the matrix F, the last $r-m$ rows are such that the first m columns of them consist of zeroes only, while in the last $r-m$ columns of them only the variables $f_{m+1}, \ldots, f_r$ occur. Putting

$$F = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

and thus

$$F^k = \begin{pmatrix} A^k & C \\ 0 & D^k \end{pmatrix},$$

where A is a matrix of m rows and columns, and the statements as to the form of F hold equally for F^k, we find that the matrix

$$\beta^{-1} = 1 + F/2! + F^2/3! + \ldots$$

is of the form

$$\beta^{-1} = \begin{pmatrix} \phi & \phi_1 \\ 0 & \phi_2 \end{pmatrix},$$

where ϕ is of m rows and columns, and ϕ_2 contains only $f_{m+1}, \ldots, f_r$;

this gives

$$\beta = \begin{pmatrix} \phi^{-1} & -\phi^{-1}\phi_1\phi_2^{-1} \\ 0 & \phi_2^{-1} \end{pmatrix};$$

so that the r infinitesimal transformations (B_s) have the forms

$$B_p = \beta_{1p}\frac{\partial}{\partial f_1} + \ldots + \beta_{mp}\frac{\partial}{\partial f_m} \quad (p = 1, \ldots, m;\ P = m+1, \ldots, r),$$

$$B_P = \beta_{1P}\frac{\partial}{\partial f_1} + \ldots + \beta_{mP}\frac{\partial}{\partial f_m} + \beta_{m+1,P}\frac{\partial}{\partial f_{m+1}} + \ldots + \beta_{r,P}\frac{\partial}{\partial f_r},$$

and herein

$$\beta_{m+1,P}, \ \ldots, \ \beta_{r,P}$$

do not contain the variables $f_1, \ldots, f_m$.

It is therefore clear why the infinitesimal transformations

$$B'_p = \beta'_{1p}\frac{\partial}{\partial f_1} + \ldots + \beta'_{mp}\frac{\partial}{\partial f_m}$$

of the corresponding invariant sub-group of the reciprocal group (B'_s) are linear functions of $B_1, \ldots, B_m$, as we previously found. The invariants of this sub-group are $f_{m+1}, \ldots, f_r$, that is, linear functions of canonical variables. Further, it is easy to see that the transformations

$$\Gamma_k = \beta_{m+1,P}\frac{\partial}{\partial f_{m+1}} + \ldots + \beta_{rP}\frac{\partial}{\partial f_r} \quad (P = m+1, \ldots, r)$$

generate a simply transitive group in $r-m$ variables $f_{m+1}, \ldots, f_r$, whose constants of structure are d_{PQR}, where each of P, Q, R is $>m$, and the variables $f_{m+1}, \ldots, f_r$ are canonical for this group (*cf.* Lie, *Transformationsgruppen*, I., p. 436).

In particular, when $m = r-1$, the coefficient β_{rr} is unity, suggesting applications to an integrable group.

7. The present section contains Darboux's proof of the necessary and sufficient conditions for a *homogeneous* contact transformation; the quite analogous general case, which is, however, deducible by a change of variables from the homogeneous case, is given in Goursat's *Partial Differential Equations of the First Order*, pp. 269 *et seq.*; the proof that the conditions are sufficient given in Goursat (p. 274) is not complete.

Let $X_1, \ldots, X_m, P_1, \ldots, P_m$ be any functions of the $2m$ variables $x_1, \ldots, x_m, p_1, \ldots, p_m$ giving rise to the identity

$$P_1 dX_1 + \ldots + P_m dX_m = p_1 dx_1 + \ldots + p_m dx_m;$$

and therefore also to the identity

$$\sum_{i=1}^{m} (\delta P_i dX_i - \delta X_i dP_i) = \sum_{i=1}^{m} (\delta p_i dx_i - \delta x_i dp_i),$$

which we write $\quad \delta P\, dX - \delta X\, dP = \delta p\, dx - \delta x\, dp,$

where $\delta x_1, \ldots, \delta x_m$, $\delta p_1, \ldots, \delta p_m$ are variations independent of $dx_1, \ldots; dx_m$, $dp_1, \ldots, dp_m$. Let ω, ω', η, η' denote respectively the matrices of the quantities

$$\omega_{ij} = \frac{\partial X_i}{\partial x_j}, \quad \omega'_{ij} = \frac{\partial X_i}{\partial p_j}, \quad \eta_{ij} = \frac{\partial P_i}{\partial x_j}, \quad \eta'_{ij} = \frac{\partial P_i}{\partial p_j},$$

each of m rows and columns; and let H, ϵ be matrices of $2m$ rows and columns

$$H = \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that in ϵ every element is zero, except

$$\epsilon_{m+i,i} = 1, \quad \epsilon_{m,m+i} = -1,$$

and
$$\epsilon^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which we denote by $\quad \epsilon^2 = -1.$

Further, let $\bar{H}$ denote the matrix

$$\bar{H} = \begin{pmatrix} \bar{\omega} & \bar{\eta} \\ \bar{\omega}' & \bar{\eta}' \end{pmatrix}$$

obtained by changing the rows of H into columns and conversely. Then the equation

$$\delta P\, dX - \delta X\, dP = \delta p\, dx - \delta x\, dp$$

is the same as

$$\epsilon H\,(dx_1, \ldots, dp_m)\, H\,(\delta x_1, \ldots, \delta p_m) = \epsilon\,(dx_1, \ldots, dp_m)(\delta x_1, \ldots \delta p_m)\,;$$

and therefore is the same as $\bar{H}\epsilon H = \epsilon$.

This shows (α) that the determinant of H is ± 1; so that $X_1, \ldots, P_m$ are necessarily independent functions of $x_1, \ldots, p_m$; in particular $X_1, \ldots, X_m$ are independent functions of $x_1, \ldots, p_m$; and thus one determinant, at least, of the matrix (ω, ω'), of m rows and $2m$ columns, is other than zero; by comparing the coefficients of dx_i and dp_i in the original equation of the contact transformation, namely,

$$\Sigma P\, dX = \Sigma p\, dx,$$

we get, however, $\qquad \bar{\omega} P = p, \quad \bar{\omega}' P = 0,$

where $\qquad P = (P_1, \ldots, P_m), \quad p = (p_1, \ldots, p_m)\,;$

the latter shows that the determinant of the matrix ω' is zero. The equation

$$\bar{H}\epsilon H = \epsilon,$$

when written $\quad \begin{pmatrix} \bar{\omega} & \bar{\eta} \\ \bar{\omega}' & \bar{\eta}' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$

and multiplied out, gives further (β), the equations expressed by

$$\bar{\omega}\eta = \bar{\eta}\omega, \quad \bar{\omega}'\eta' = \bar{\eta}'\omega', \quad \bar{\omega}\eta' - \bar{\eta}\omega' = 1 = \bar{\eta}'\omega - \bar{\omega}'\eta\,;$$

which are of the same form as the so-called Weierstrassian relations connecting the periods of an Abelian function; while also we deduce (γ) from

$$\bar{H}\epsilon H = \epsilon,$$

that $\bar{H}\epsilon H\epsilon = -1, \quad \epsilon H\epsilon = -(\bar{H})^{-1}, \quad H\epsilon = -\epsilon^{-1}(\bar{H})^{-1} = \epsilon(\bar{H})^{-1},$

and hence $\qquad H\epsilon\bar{H} = \epsilon,$

which is the same as

$$\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{\omega} & \bar{\eta} \\ \bar{\omega}' & \bar{\eta}' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and this gives, on multiplying out, the relations

$$\omega\bar{\omega}' = \omega'\bar{\omega}, \quad \eta\bar{\eta}' = \eta'\bar{\eta}, \quad \omega\bar{\eta}' - \omega'\bar{\eta} = 1 = \eta'\bar{\omega} - \eta\bar{\omega}',$$

identical in form with the so-called Riemann relations for an Abelian function. Putting

$$(\phi, \psi) = \sum_{r=1}^{m} \left(\frac{\partial\phi}{\partial p_i}\frac{\partial\psi}{\partial x_i} - \frac{\partial\phi}{\partial x_i}\frac{\partial\psi}{\partial p_i}\right),$$

these are the equations

$$(P_i X_i) = 1, \quad (P_i X_j) = (X_i X_j) = (P_i P_j) = 0$$

$$(i \neq j, \; i, j = 1, \; \ldots, \; n).$$

Lastly, as $\bar{\omega}P = p, \quad \bar{\omega}'P = 0,$

the equation $\bar{H}\epsilon H = \epsilon$

gives (δ) that

$$\omega' p = (\omega'\bar{\omega} - \omega\bar{\omega}')\, P = 0,$$

$$\eta' p = (\eta'\bar{\omega} - \eta\bar{\omega}')\, P = P,$$

or

$$\omega' p = 0, \quad \eta' p = P;$$

so that each of $X_1, \ldots, X_m$ is a homogeneous function of $p_1, \ldots, p_m$ of zero degree, and each of $P_1, \ldots, P_m$ is a homogeneous function of $p_1, \ldots, p_m$ of degree unity.

Conversely, if $X_1, \ldots, X_m$ be independent functions of the $2m$ variables $x_1, \ldots, x_m, p_1, \ldots, p_m$, which are homogeneous of zero degree in $p_1, \ldots, p_m$, and satisfy the $\frac{1}{2}m\,(m-1)$ equations

$$(X_i X_k) = 0,$$

it can be shown, as below, that m functions $P_1, \ldots, P_m$ can be found by the solution of linear algebraic equations, such that

$$\Sigma P_i dX_i = \Sigma p_i dx_i,$$

and that these functions $P_1, \ldots, P_m$ are homogeneous in $p_1, \ldots, p_m$ of degree unity; so that the conditions specified are sufficient to determine a homogeneous contact transformation.

First, if (ω, ω') be a matrix of m rows and $2m$ columns of which not every determinant of order m is zero, which is such that

$$\omega\bar{\omega}' - \omega'\bar{\omega} = 0;$$

and if $m-k$ be the rank of the matrix ω', a determinant of m rows and columns from (ω, ω') can be found which does not vanish, containing $m-k$ columns of ω' and the k *complementary* columns of ω. This can be proved from the theory of determinants, or simply by proving that the m linear equations

$$u_i = \omega_{i1} x_1 + \ldots + \omega_{im} x_m + \omega'_{i1} y_1 + \ldots + \omega'_{im} y_m$$

can be solved for k of $x_1, \ldots, x_m$ and the complementary $m-k$ of $y_1, \ldots, y_m$.

We have now, in the previous notation,

$$\omega' p = 0, \quad \omega\omega' - \omega'\bar{\omega} = 0,$$

and we show that the $2m$ linear equations for $P_1, \ldots, P_m$,

$$\bar{\omega} P - p = 0, \quad \bar{\omega}' P = 0,$$

have a solution. By the conditions we know that m of these equations are linearly independent and have a solution, and as, from

$$\omega' p = 0,$$

it is known that

$$|\omega'| = 0,$$

these m independent equations are not the equations

$$\omega' P = 0.$$

Denoting the left sides of the equations by $L_1, \ldots, L'_m$, so that

$$L_i = \omega_{1i} P_1 + \ldots + \omega_{mi} P_m - p_m, \quad L'_i = \omega'_{1i} P_1 + \ldots + \omega'_{mi} P_m,$$

we immediately prove

$$\omega'(L_1, \ldots, L_m) - \omega(L'_1, \ldots, L'_m) = 0,$$

namely, the left sides are connected by m equations

$$\omega'_{j1} L_1 + \ldots + \omega'_{jm} L_m - \omega_{j1} L'_1 - \ldots - \omega_{jm} L'_m = 0,$$

and it is to be shown that, if the values of $P_1, \ldots, P_m$ which make a certain m of the quantities $L_1, \ldots, L'_m$ vanish be substituted herein, the determinant of the coefficients of the others will not be zero; so that these others are necessarily also zero.

Suppose, then, $P_1, \ldots, P_m$ can be chosen to make

$$L_1 = 0, \; \ldots, \; L_k = 0, \quad L'_{k+1} = 0, \; \ldots, \; L'_m = 0;$$

this is in accordance with the lemma put down above. Then the determinant of the quantities L which remain is

$$| \omega'_{j,k+1}, \ldots, \omega'_{jm}, \omega_{j1}, \ldots, \omega_{jk} |,$$

which is exactly the determinant assumed not to be zero. The result would not follow unless the second suffixes of the ω'_{jk} in this determinant were the complementary of the second suffixes of the ω_{jk}.

Thus the equations

$$(X_i X_k) = 0, \quad \Sigma p_i \frac{\partial X}{\partial p_i} = 0, \quad \left| \frac{\partial (X_1, \ldots, X_m)}{\partial (x_1, \ldots, x_m, p_1, \ldots, p_m)} \right| \neq 0$$

are sufficient uniquely to determine a homogeneous contact transformation.

The following presents and exchanges have been received during the Recess:—

"Educational Times," July-Oct., 1901.

"Indian Engineering," Vol. xxix., May 25–June 22, Nos. 21–25; Vol. xxx., July 6–Sept. 7, Nos. 1–10; 1901.

"Kansas University Quarterly," Vol. ix., Nos. 1–4, 1900; Vol. x., No. 1, 1901.

"Mathematical Gazette," Vol. ii., No. 28; 1901.

"Annals of Mathematics," Ser. 2, Vol. ii., No. 4; July, 1901.

"John Crerar Library—Sixth Annual Report for 1900"; Chicago, 1901.

"Institution of Civil Engineers—List of Members," 1901.

"Henry Perigal: his Life and Works," 8vo; London. Compiled and presented by F. Perigal.

Biddle, D.—"Mathematical Questions and Solutions," Vol. lxxv; 1901.

Kantor, S.—"I Numeri razionali in Geometria," 8vo; Torino, 1901.

Stoney, G. J.—"Survey of that part of the range of Nature's Operations which Man is Competent to Study," 8vo; Washington, 1901.

Fick, R.—"Ueber die Bewegungen in den Handgelenken," 8vo; Leipzig, 1901.

Fischer, O.—"Der Gang des Menschen," 8vo; Leipzig, 1901.

Garten, S.—"Ueber rhythmische, elektrische Vorgänge im quergestreiften Skelettmuskel," 8vo; Leipzig, 1901.

Offprints from the "Memorie dello Reale Istituto Lombardo di Scienze e Lettere":—

Monti, R.—"La Rigenerazione nelle Planarie marine," 4to; Milano, 1900.

Scarenzio, A.—"Il Processo di Rinoplasteon a Lembo frontale duplicato di Blasius," 4to; Milano, 1900.

Monti, A.—"Su gli Scheletri di alcune Scimmie Rachitiche," 4to; Milano, 1900.

Tommasi, A.—"Contribuzione alla Paleontologia della Valle del Dezzo," 4to; Milano, 1901.

Artini, E., e Melzi, G.—"Ricerche petrografiche e geologiche sulla Valsesia," 4to; Milano, 1900.

"Opere matematiche di Francesco Brioschi," Tomo I., 4to; Milano, 1901. With the "Omaggio del Comitato par le Onoranzo a Francesco Brioschi."

Dickson, L. E.—"Linear Groups, with an Exposition of the Galois Field Theory," 8vo; Leipzig, 1901. From the Author.

"Cambridge University Library Report for the year 1900."

From the "Scientia" Series, presented by the editor, M. C. Naud:—

"Production et Emploi des Courants Alternatifs," par L. Barbillon, No. 11.

"La Série de Taylor et son Prolongement Analytique," par Jacques Hadamard, No. 12.

From the same editor:—

"L'Enseignement Mathématique," Nos. for July 15 and September 15, 1901.

"Reciprocally related Figures and the Property of Equianharmonicity," by Jamshedji Edalji, B.A. (Ahmedabad, 1901). From the Author.

"Sulla Genesi Combinatoria dell' Aritmetica," by A. Capelli (offprint from the "Giornale di Matematiche"); Napoli, 1901.

"Proceedings of the Royal Society," Vol. LXVIII., Nos. 446–450.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXV., Hefte 7–10; Leipzig, 1901.

"Rendiconti del Circolo Matematico di Palermo," Tomo XV., Fasc. 3, 4; 1901.

"Bulletin de la Société Mathématique de France," Tome XXIX., Fasc. 3; Paris, 1901.

"Annales de la Faculté des Sciences de Toulouse," Tome II., Fasc. 3, 4; Paris, 1900.

"Bulletin of the American Mathematical Society," Vol. VII., Nos. 9, 10; New York, 1901.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XIV., No. 4; Coimbra, 1901.

"Reale Istituto Lombardo—Rendiconti," Vol. XXXIII.; Milano.

"Bulletin des Sciences Mathématiques," "Table des Matières," Tome XXIV., 1900; Tome XXV., Avr.–Juin, 1901; Paris.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. VII., Fasc. 5, 6, 7; Napoli, 1901.

"Journal für die reine und angewandte Mathematik," Band CXXIII., Heft 4; Berlin, 1901.

"Annali di Matematica," Tomo VI., Fasc. 1, 2, 3; Milano, 1901.

"Sitzungsberichte der Physikalisch-medicinischen Societät in Erlangen," Heft 32; 1900.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. X., Sem. 1, Fasc. 11, 12; Sem. 2, Fasc. 1–5; "Rendiconto (della Reale Accademia dei Lincei, Anno 298, 1901) dell' Adunanza solenne del 2 Giugno 1901, onorata dalla presenza di S.M. il Re"; Roma, 1901.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," Bd. LIII., 1, 2, 3; 1901.

"Revue Semestrielle des Publications Mathématiques," Tome IX., Pte. 2; Amsterdam, 1901.

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It was ordered by the Council (Feb. 13, 1896), "That separate parts of the *Proceedings* may be obtained from the Publisher at 1*s.* 6*d.* per sheet, when the number in stock exceeds 100."

"That Members be allowed to complete Sets at trade price" [*i.e.*, at 1*s.* 4*d.* per sheet].

It was further ordered by the Council (March 9, 1899), "That the Society allow 5 per cent. extra discount to each purchaser of a *complete* set of *Proceedings up to the last completed volume.*"

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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

EDITED BY R. TUCKER AND A. E. H. LOVE.

VOL. XXXIV.—NOS. 772—776

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contributions which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of ten guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the year 1902 are the *second* Thursdays in January, February, March, April, May, June, November, and December.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained.

Communications for the Secretaries may be forwarded to them at the followin addresses:—

London Mathematical Society, 22 Albemarle Street, W. } R. TUCKER.
24 Hillmarton Road, West Holloway, N. }
34 St. Margaret's Road, Oxford.—A. E. H. LOVE.

"Journal of the Institute of Actuaries," Vol. XXXVI., Pt. 2; 1901.

"Memorie della Regia Accademia in Modena," Ser. 3, Vol. II.; 1900.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zurich," Vol. XLVI., Hefte 1, 2; 1901.

"Nieuw Archief voor Wiskunde," Deel V., St. 2; Amsterdam, 1901.

"Proceedings of the Physical Society," Vol. XVII., Pt. 6; London, July, 1901.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," 23–38; 1901.

"Proceedings of the Cambridge Philosophical Society," Vol. XI., Pt. 3; 1901.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Geschäftliche Mitteilungen, Heft 1, 1901; Math.-phys. Klasse, Heft 1, 1901.

"Jahrbuch über die Fortschritte der Mathematik," Band XXX., Jahrgang 1900, Hefte 1, 2; 1901.

"Transactions of the Royal Irish Academy," Vol. XXXI., Pts. 9–12; Dublin, 1900, 1901.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome IV., Livr. 3; Harlem, 1901.

"Transactions of the Canadian Institute," Vol. VII., Pt. 1, No. 13; Toronto, 1901.

"Supplemento al Periodico di Matematica," Anno IV., Fasc. 8 and 9; Livorno, 1901.

"Periodico di Matematica," Anno XVI., Fasc. 6; Anno XVII., Fasc. 1; Livorno, 1901.

"Proceedings of the American Philosophical Society," Vol. XL., No. 165; January, 1901.

"The American Journal of Mathematics," Vol. XXIII., 3, 4; New York, 1901.

APPENDIX (*b*).

(Session 1900–1901.)

The Council are indebted to Prof. Elliott for the following notice:—

Mr. Charles Edward Bickmore, M.A., who died at his residence in Oxford, on April 30th, 1901, had been a member of the Society since February 11th, 1875. He was eldest son of the Rev. Charles Bickmore, D.D., of Berkswell Hall, Warwickshire, and Leamington. His school was Westminster, where he was Captain in 1866. He proceeded to Christ Church, Oxford, as a Westminster Junior Student,

and at Oxford obtained a First Class in Mathematics and Second Class in Classics from Moderators, and a First Class in Mathematics at the Final Honours Examination. In 1872 he was elected to a Fellowship at New College, which he vacated on marriage in 1885. From 1872 till 1878 he served his College as Mathematical Lecturer. He also lectured for Pembroke and Keble Colleges. He was examiner within the University on a number of occasions; and did much examining work externally.

Mr. Bickmore was an enthusiastic student of higher arithmetic and the theory of numbers, and was active as a researcher on such questions as factorization, and on the use of periodic continued fractions. Much, perhaps most, of his original work was unhappily left in an unfinished state. But he published new results, and collected much of interest which was scattered and little known, in a number of papers in the *Messenger of Mathematics* and the *Nouvelles Annales.* He also produced a considerable continuation of Degen's Tables connected with the Pellian equation, which is incorporated in a report to the British Association (1893). He never submitted a paper for our own *Proceedings*, though he not infrequently took part in our discussions when an arithmetical paper was before us. At the time of his decease he was President of the Oxford Mathematical Society, a non-publishing society to which he had made very many communications.

Lt.-Col. Cunningham supplies the following references:—

Mathematical Papers by the late C. E. Bickmore.

1. In *Messenger of Mathematics*, Vol. xxv., 1895–6, pp. 1–44, "On the Numerical Factors of $(a^n - 1)$."
2. In same periodical, Vol. xxvi., 1896–7, pp. 1–38, same title (second notice).
3. In *Nouvelles Annales de Mathématiques*, Série 3, Vol. xv., 1896, pp. 222–227, "Sur les Fractions Décimales Périodiques."
4. In *British Association Report*, 1893, pp. 73–120, "Tables connected with the Pellian Equation" ($y^2 = ax^2 + 1$, from $a = 1001$ to 1500).
5. Extensive Tables of the *complex* 8-ic factors of primes of form $(8\varpi + 1)$, up to about 20,000 (in MS.).

CORRIGENDA.

Page 47, second table, *for* $G_2^2 = G_1 + 2G_4$ $\quad G_3^2 = G_1 + 2G_4$,
read $G_2^2 = G_1 + G_2 + G_5$ $\quad G_3^2 = G_1 + G_3 + G_6$.

THIRTY-EIGHTH SESSION, 1901–1902

(since the Formation of the Society, January 16th, 1865).

November 14*th*, 1901.

The Eighth Annual General Meeting of The London Mathematical Society, as incorporated under the Companies Act, 1867, on October 23rd, 1894, held at 22 Albemarle Street, W.

Dr. E. W. HOBSON, F.R.S., President, in the Chair.

Eighteen members and a visitor present.

The President spoke briefly upon the loss the Society had sustained through the death of Mr. J. Hamblin Smith, M.A., who was elected a member, December 8th, 1870.

Mr. Robert John Dallas, B.A., was elected a member, and Prof. Alfred Lodge was admitted into the Society.

The Treasurer read his report, the reception of which was moved by Major MacMahon, seconded by Prof. A. Lodge, and carried unanimously.

The President then moved that, if it was the pleasure of the meeting, and if Mr. Gallop was willing to act, Mr. Gallop be again appointed the Society's Auditor. The motion was carried unanimously.

Mr. Tucker stated that the Secretaries had heard of the deaths of three members during the past Session, viz., Prof. C. Hermite, elected an honorary member, December 14th, 1871; Mr. C. E. Bickmore, who was elected February 11th, 1875; and Mr. J. H. Smith. He further announced that an exchange had been arranged with the Institution of Naval Architects, and that the Council had sanctioned a five years' subscription for the three annual volumes of the International Catalogue, referring to "Physics," "Mathematics," and "Mechanics."

The number of members at the commencement of the session 1900–1901 was 252. During the session 5 members resigned, 2 died, the name of one was struck off, and 13 new members were elected.

One member having been elected November 14th, 1901, there are now on the Society's list 258 members.

At the President's request Dr. R. Bryant and Prof. Lodge acted as Scrutineers, when the ballot for the election of the Council of the new session was taken. The following gentlemen were declared by them to be duly elected to serve on the Council:—President: Dr. Hobson; Vice-Presidents: Prof. Burnside and Major MacMahon; Treasurer: Dr. J. Larmor; Hon Secs.: Mr. R. Tucker and Prof. Love. Other members: Messrs. J. E. Campbell, Lt.-Col. Cunningham, Prof. Elliott, Dr. Glaisher, Prof. Hill, Mr. H. M. Macdonald, Prof. Rogers, Mr. A. E. Western, Mr. E. T. Whittaker, and Mr. Alfred Young.

Dr. Larmor propounded a query regarding the recent behaviour of Nova Persei, with a view to getting the opinions of the astronomical members present. The President, Dr. Glaisher, Messrs. Hargreaves and Hough, and Lt.-Col. Cunningham joined in a discussion on the subject.

Prof. Love communicated two papers by Mr. J. H. Michell, (1) "On the Inversion of Plane Stress," and (2) "On the Theory of Hele-Shaw's experiments on Fluid Motion"; the latter paper gave rise to some discussion.

Mr. Whittaker read a paper "On the Solution of Dynamical Problems in terms of Trigonometrical Series," which also gave rise to some discussion.

The following papers were communicated by reading the titles:—

Linear Groups in an Infinite Field: Dr. L. E. Dickson.

Note on the Algebraic Properties of Pfaffians: Mr. J. Brill.

On Burmann's Theorem: Prof. A. C. Dixon.

The Puiseux Diagram and Differential Equations: Mr. R. W. H. T. Hudson.

Determination of all the Groups of Order 168: Dr. G. A. Miller.

An Outline of a Theory of Divergent Integrals: Mr. G. H. Hardy.

Limits of Logical Statements: Mr. H. MacColl.

Addition Theorems for Hyperelliptic Integrals: Mr. A. L. Dixon.

On the Representation of a Group of Finite Order as a Permutation Group, and on the Composition of Permutation Groups: Prof. W. Burnside.

Note on Clebsch's Transformation of the Equations of Hydrodynamics: Mr. T. Stuart.

Linear Null-Systems of Binary Forms: Mr. J. H. Grace.

The following presents were made to the Library:—

"Educational Times," November, 1901.

"Indian Engineering," Vol. xxx., Nos. 11-16, Sept. 14-Oct. 19, 1901.

"Mathematical Gazette," Vol. II., No. 29; 1901.

Rayet, G.—"Observations Pluviométriques et Thermométriques de la Gironde de Juin 1899 à Mai 1900," 8vo; Bordeaux, 1900.

Scarpini, G.—"Tavole Numeriche di Topografia," 8vo; Torino, 1901.

Tycho Brahe.—"Operum Primitias de Nova Stella summi civis memor denuo edidit Regia Societas Scientiarum Danica," 4to; Hauniæ, 1901.

"Annals of Mathematics," Series 2, Vol. III., No. 1; 1901.

"Journal de l'Ecole Polytechnique," Série 2, Cah. 5, 6; Paris, 1900-1.

Huygens, C.—"Œuvres Complètes," Tome IX., 4to; La Haye, 1901.

Basset, A. B.—"Elementary Treatise on Cubic and Quartic Curves," 8vo; Cambridge, 1901. (From the Author.)

The following exchanges were received:—

"Transactions of the Royal Society," Vol. CXCVI.; 1901.

"Proceedings of the Royal Society," Vol. LXIX., No. 451; 1901.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxv., Heft 11; Leipzig, 1901.

"Annales de la Faculté des Sciences de Toulouse," Série 2, Tome III., Fasc. 1; Paris, 1901.

"Transactions of the American Mathematical Society," Vol. II., No. 4; New York, 1901.

"Bulletin of the American Mathematical Society," Vol. VIII., No. 1; New York, 1901.

"Proceedings of the American Philosophical Society," Vol. XL., No. 166; 1901.

"Monatshefte für Mathematik und Physik," Jahrgang XII., Pt. 4; Wien, 1901.

"Bulletin des Sciences Mathématiques," Tome xxv., Juillet, Août, Sep.; Paris, 1901.

"Journal für die reine und angewandte Mathematik," Bd. CXXIV., Heft 1; Berlin, 1901.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. x., Sem. 2, Fasc. 7, 8; Roma, 1901.

"Annuaire de l'Académie Royal de Belgique," 1900-1901; Bruxelles, 1901.

"Bulletin de l'Académie Royale de Belgique," Classe de Sciences, 1899-1900; Bruxelles.

"Mémoires de la Société des Sciences Physiques et Naturelles," Série 5, Tome v., Cah. 2; Bordeaux, 1901.

"Procès-Verbaux des Séances de la Société des Sciences Physiques et Naturelles," Année 1899-1900; Bordeaux, 1900.

"Periodico di Matematica," Anno xvii., Fasc. 2; Livorno, 1901.
"Journal of the Institute of Actuaries," Vol. xxxvi., Pt. 3; 1901.
"Wiskundige Opgaven," Deel viii., St. 4; Amsterdam, 1901.
"Prace Matematyczno-Fizyczne," Tom. xii.; Warsaw, 1901.

The Inversion of Plane Stress. By J. H. Michell.

Received October 26th, 1901. Read November 14th, 1901.

1. A process of inversion can be applied to the stress-function* ψ, satisfying the differential equation

$$\nabla^4\psi = 0,$$

similar to that employed in electrical theory to transform the potential.

The stress in the inverted solution is connected with that in the original solution in a simple manner. In particular, it may be noted that the lines of principal stress invert into lines of principal stress and uniform normal stress into uniform normal stress; thus a boundary free from stress becomes, in general, a boundary under uniform normal stress.

This process of inversion is illustrated by the reduction of circular to straight boundaries. In particular, it is shown that the solution for uniform normal stress over any portions of a circular boundary can be expressed in simple finite terms.

Applications to mathematically allied problems, involving extended analysis, are held over for subsequent papers. The extension to three-dimensional problems is merely noted.

The Inverted Stress-Function.

2. The stress-function ψ, in the absence of volume-forces, satisfies the differential equation

$$\nabla^4\psi = 0,$$

---

* The notation and the properties of ψ assumed are taken from two earlier papers, *Proc. Lond. Math. Soc.*, Vol. xxxi., p. 100; Vol. xxxii., p. 35.

and the polar elements of stress are expressed by

$$\sigma_r \qquad P = \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\psi}{\partial r},$$

$$\sigma_\theta \qquad Q = \frac{\partial^2\psi}{\partial r^2},$$

$$\tau_{r\theta} \qquad U = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right).$$

Let the plane be inverted with respect to the origin of the polar coordinates (r, θ), the point (r, θ) going to (ρ, θ). For simplicity take unity as the radius of inversion; so that

$$r\rho = 1.$$

The function ψ', $= \frac{\psi}{r^2} = \psi\rho^2$, is a stress-function satisfying $\nabla^4\psi' = 0$ in the inverse plane. To prove this, introduce conjugate coordinates

$$z = x + \iota y, \quad z' = x - \iota y.$$

The general solution of $\qquad \nabla^4\psi = 0$

is $$\psi = z'f(z) + zg(z') + F(z) + G(z'),$$

or $$\frac{\psi}{zz'} = \frac{f(z)}{z} + \frac{g(z')}{z'} + \frac{F(z)}{zz'} + \frac{G(z')}{zz'}.$$

Let (ξ, η) be the inverse of (x, y), and put

$$\zeta = \xi + \iota\eta, \quad \zeta' = \xi - \iota\eta; \qquad \zeta = \frac{1}{z}$$

so that $$\zeta z' = \zeta' z = 1.$$

Then $$\frac{\psi}{zz'} = f_1(\zeta') + g_1(\zeta) + \zeta F_1(\zeta') + \zeta' G_1(\zeta),$$

where $$f_1(\zeta') = \frac{f(\zeta')}{\zeta'},$$

and so on. This shows that ψ', $= \frac{\psi}{zz'} = \frac{\psi}{r^2}$, satisfies $\nabla^4\psi' = 0$ in the inverse plane. The relation between ψ and ψ' is plainly a reciprocal one.

Comparison of Stresses.

3. Denote by P', Q', U' the polar elements of stress in the inverse plane; so that

$$P' = \frac{1}{\rho^2}\frac{\partial^2\psi'}{\partial\theta^2} + \frac{1}{\rho}\frac{\partial\psi'}{\partial\rho},$$

$$Q' = \frac{\partial^2\psi'}{\partial\rho^2},$$

$$U' = -\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi'}{\partial\theta}\right).$$

Now

$$\frac{\partial\psi'}{\partial\rho} = -r^2\frac{\partial}{\partial r}\frac{\psi}{r^2},$$

$$\frac{\partial^2\psi'}{\partial\rho^2} = r^2\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\frac{\psi}{r^2}$$

$$= r^4\frac{\partial^2\psi}{\partial r^2} - 2r^3\frac{\partial}{\partial r}\frac{\psi}{r};$$

therefore

$$\frac{1}{\rho^2}\frac{\partial^2\psi'}{\partial\theta^2} + \frac{1}{\rho}\frac{\partial\psi'}{\partial\rho} = r^2\left(\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right) + 2\left(\psi - r\frac{\partial\psi}{\partial r}\right),$$

$$\frac{\partial^2\psi'}{\partial\rho^2} = r^2\frac{\partial^2\psi}{\partial r^2} + 2\left(\psi - r\frac{\partial\psi}{\partial r}\right),$$

$$\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi'}{\partial\theta}\right) = -r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right);$$

so that

$$P' = r^2P + 2M,$$

$$Q' = r^2Q + 2M,$$

$$U' = -r^2U,$$

where

$$M = \psi - r\frac{\partial\psi}{\partial r}.$$

Hence the stress in the inverse plane is compounded of an isotropic* tension $2M$ and a stress which is heterochirally similar to that at

* A word is wanted to describe the centro-symmetrical "fluid" or "hydrostatic" pressure or tension. "Isotropic" is here used provisionally.

the corresponding point of the original plane, and of r^2 times the intensity.

Otherwise stated, the total stress across a line-element $\delta s'$ in the inverse plane is compounded of a normal tension $2M\delta s'$ and a stress which is equal to that across the corresponding line element δs in the original plane and in the corresponding direction. This follows because the line-elements are in the ratio $1 : r^2$.

In particular, if the stress is normal across δs, it is also normal across $\delta s'$, and therefore lines of principal stress invert into lines of principal stress.

Interpretation of M.

4. The quantity M at a point (r, θ) differs by a constant merely from the moment of the stress across a line from a fixed point to (r, θ), taken about the origin of inversion. To prove this, let s be the arc of a curve from a fixed point to (r, θ). The moment about the origin of inversion of the stresses across this arc is

$$\begin{aligned}\int (r^2 U\,d\theta - rQ\,dr) &= -\int \left(r\frac{\partial^2\psi}{\partial r^2}\,dr + r^2\frac{\partial}{\partial r}\,\frac{1}{r}\,\frac{\partial\psi}{\partial\theta}\,d\theta\right)\\ &= -\int \left\{\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)dr + \frac{\partial}{\partial\theta}\left(r\frac{\partial\psi}{\partial r}\right)d\theta\right\}\\ &\quad + \int\left(\frac{\partial\psi}{\partial r}\,dr + \frac{\partial\psi}{\partial\theta}\,d\theta\right)\\ &= \left[\psi - r\frac{\partial\psi}{\partial r}\right].\end{aligned}$$

Hence the proposition. The connexion, expressed by the equation

$$\psi' - \rho\frac{\partial\psi'}{\partial\rho} = \frac{-\psi + r\dfrac{\partial\psi}{\partial r}}{r^2},$$

between the moments of the stresses across corresponding lines in the two planes should be noted.

Uniform Normal Tension.

5. If the stress across s is a uniform normal tension T, its moment about the origin of inversion is

$$-\int Tr\frac{\partial r}{\partial s}\,ds = -\tfrac{1}{2}Tr^2 + C.$$

Hence the stress at the corresponding point across the inverted curve s', which is normal and equal to Tr'^2+2M, is also uniform. In particular, a boundary free from stress inverts into a boundary under uniform normal tension (or pressure). In the case of singly-connected regions this uniform normal tension is immediately removable by the superposition of a uniform isotropic pressure throughout. In the inversion of multiply-connected regions it must be remembered that the condition for single-valued displacements will not in general be simultaneously satisfied by a solution and its inverse.

Comparison of Concentrated Forces.

6. If the stress becomes infinite at some point, not the centre of inversion, in the original plane, let that point be surrounded by a very small circuit s. Corresponding to this there is a very small circuit s' in the inverse plane. If, as will be supposed, the moment of the forces across any arc, about the origin of inversion, is finite, the term $2M$ in the inverse stress is finite, and therefore adds nothing to the resultant stress across an infinitesimal circuit. Omitting this term, the stresses on corresponding elements of s, s' become equal and in corresponding directions; hence the resultant stresses across s, s' are equal and in corresponding directions. Concentrated forces, therefore, invert into concentrated forces of the same magnitude and in corresponding directions. As in the last section, it must be remembered that, if concentrated forces act in the interior of the body, the inverse solution will not in general satisfy the conditions for single-valued displacement.

When the original region extends to infinity there may be a concentrated force or couple, or both, at the origin of inversion in the inverse solution corresponding to the stress at infinity in the original solution. This force is readily determined in any given case by the form of ψ' in the neighbourhood of the origin, or by the conditions of equilibrium of the body as a whole. A general discussion is hardly necessary.

The Circular Plate or Cylinder.

7. The solution $\psi = r'\theta' \sin\theta' = y\theta'$

gives a simple radial distribution* of stress at the origin O' of the

* *Proc. Lond. Math. Soc.*, Vol. XXXII., pp. 44, 54.

polar coordinates (r', θ'). If we take a straight line through this origin as the boundary of an otherwise unlimited plate, the solution corresponds to a concentrated force at O' in the negative direction of the initial line and of magnitude π.

Invert with respect to a point on the initial line outside the plate. The straight boundary inverts into a circle on which there are *two* balancing concentrated forces at the ends of a chord, the second concentrated force being at the origin O of inversion. There is also a uniform normal tension over the circle to be removed by the superposition of a uniform isotropic pressure throughout. This is equivalent to the solution given by Hertz and discussed geometrically in a previous paper.*

In the original solution the lines of stress are the circles around O' and their radii. Therefore in the inverse solution the lines of stress are the circles through the points of application O, O'' of the forces and their orthogonals. This result is not affected by the superposition of the isotropic stress. Further, there is no stress across the radii from O' in the original solution, and therefore the stress is uniform on each of the circles through O, O''.

For the analytical form of the solution, let (ρ', ϑ') be polar coordinates at O'', the inverse of O'. Then

$$\theta' = \pi + \theta - \vartheta', \quad r' = c\rho'/\rho,$$

where $OO' = c$.

Hence
$$\begin{aligned} \psi' &= \rho^2 r'\theta' \sin\theta' \\ &= \rho^2 y\,(\pi+\theta-\vartheta') \\ &= \eta\,(\pi+\theta-\vartheta') \\ &= \pi\eta + \rho\theta\sin\theta - \rho'\vartheta'\sin\vartheta'. \end{aligned}$$

Thus, omitting the insignificant term $\pi\eta$, ψ' is the difference of the stress-functions corresponding to two simple radial distributions at O, O'', a result already known.

Uniform Pressure on a Straight Boundary.

8. Let a uniform normal pressure be applied along a finite length of a straight boundary of an otherwise unlimited plate. The form of ψ is found by integration from the solution for concentrated force.

---

* *Loc. cit.*

It is sufficient to write down and verify the result. Let AB be the length on which the pressure is applied. Then, with the proper units,

$$\psi = r^2\theta - r'^2\theta',$$

where (r, θ), (r', θ') are polar coordinates at A, B and $\overline{AB}$ is the initial line. For verification, observe that the polar elements of stress, origin A, corresponding to $\psi = r^2\theta$ are

$$P = 2\theta,$$

$$Q = 2\theta,$$

$$U = -1,$$

and to $-r'^2\theta'$, origin B, are

$$P' = -2\theta',$$

$$Q' = -2\theta',$$

$$U' = 1.$$

Therefore along the initial line the tangential stress is $U+U'$, $= 0$, throughout; along AB there is a uniform normal pressure $2(\theta'-\theta)$, $= 2\pi$, and the rest of the boundary is free from stress. Finally, the stress vanishes at infinity.

To find the principal stresses at any point, it is convenient to use the formulæ for the transformation of axes in a different form from that usually employed. Let accented letters denote the stress elements for axes in angular advance θ of the original axes. Then

$$P'+Q' = P+Q,$$

$$P'-Q'-2\iota U' = e^{2\iota\theta}(P-Q-2\iota U),$$

as may be readily deduced from the usual formulæ. Now, let $\overline{P}$, $\overline{Q}$, $\overline{U}$ be the elements of stress, in the solution found, referred to the bisectors of the angle APB $(=\alpha)$, at the point $P(r, \theta)$, (r', θ'). Then, by the formulæ quoted,

$$\overline{P}+\overline{Q} = P+Q+P'+Q' = 4(\theta-\theta') = -4\alpha,$$

$$\begin{aligned}\overline{P}-\overline{Q}-2\iota\overline{U} &= e^{\iota\alpha}(P-Q-2\iota U)+e^{-\iota\alpha}(P'-Q'-2\iota U') \\ &= 2\iota(e^{\iota\alpha}-e^{-\iota\alpha}) \\ &= -4\sin\alpha;\end{aligned}$$

therefore

$$\bar{P} = -2\,(\alpha + \sin\alpha),$$

$$\bar{Q} = -2\,(\alpha - \sin\alpha),$$

$$\bar{U} = 0.$$

Thus the principal axes of stress bisect the angle APB, and the principal lines of stress are therefore the confocal conics having A, B as foci. It further appears that the stress is uniform along each circle through A, B.

Uniform Pressure on Part of a Circular Plate.

9. Invert the solution of the last section with respect to a point outside the plate, and on the line bisecting AB at right angles. The boundary is now a circle under different uniform normal pressures on the two parts into which the inverses A', B' of A, B divide the circle, and with a balancing concentrated force at the origin O of inversion (which is on the diameter bisecting the arc $A'B'$). The part $A'OB'$ of the boundary may be freed from the uniform normal pressure on it by the superposition of a uniform isotropic tension throughout, leaving a normal pressure over the arc $A'B'$ balanced by the force at O.

The lines of principal stress are the inverses of the confocal conics of the last section with respect to O.

The concentrated force at O can similarly be replaced by a uniform normal pressure over another arc bisected by the diameter through O. The stress-function due to such uniform distributions of normal pressure is therefore expressible in finite terms. The analytical form of the solutions is readily found. With the notation of § 7, a term of the form

$$\psi = r'^2\theta'$$

inverts into

$$\psi' = c^2\rho'^2\,(\pi + \theta - \vartheta')$$

$$= \pi c^2\rho'^2 - c^2\rho'^2\vartheta' + c^2\,(\rho^2 + c^2 - 2c'\xi)\,\theta,$$

where

$$cc' = 1.$$

Thus ψ' is built up of elementary solutions of the forms (a) ρ^2 (uniform isotropic tension), (b) $\xi\theta$ or $\rho\theta\cos\theta$ (simple radial distribution), (c) $\rho^2\theta$ (as in § 8), (d) θ (symmetrical shear around the origin). The complete solution for any case is therefore composed of terms of these forms, but terms of type (d) do not occur in the final result.

A Uniformly Loaded Beam.

10. A final example, illustrating the transmission of stress in a loaded beam supported at its ends, is perhaps worth noting. Let a plate in the form of a segment of a circle be supported at its ends and uniformly loaded along the chord. The solution can be found by starting with a single force at the angle of an infinite plate bounded by two straight lines at an angle equal to the angles of the segment. A simple radial distribution at the angle solves this subsidiary problem. Let the force be at right angles to one of the straight boundaries, and invert with respect to a point on this straight boundary produced. The inverse solution gives a segmental plate acted on by two concentrated forces at the angles perpendicular to the chord of the segment, and by different uniform normal stresses over the chord and arc. Removing the uniform normal stress over the arc by the superposition of a uniform isotropic stress throughout, the required solution is reached.

Since the lines of principal stress in the subsidiary problem are the concentric circles around the angle and their radii, those in the beam are the circles through the angles of the segment and their orthogonals. Also, since there is no stress across the radii from the angle in the subsidiary problem, the stress across each circle through the angles of the segment is uniform. It is easily shown that the difference of the principal stresses along any one of these circles is inversely proportional to the distance from the loaded chord.

Extension to Three Dimensions.

11. The corresponding law of inversion of a solution of

$$\nabla^4\psi = 0$$

in three dimensions is $$\psi' = \frac{\psi}{r}.$$

The author has not yet made a detailed examination of the application of this inversion.

Note on the Algebraic Properties of Pfaffians. By J. BRILL, M.A.
Received September 17th, 1901. Read November 14th, 1901.

1. My object is, in the first instance, to obtain a generalization of the ordinary theorem for the expansion of a Pfaffian, viz.,

$$[123 \dots (2m)] = [12][345 \dots (2m)] - [13][245 \dots (2m)] + \dots$$
$$\dots + (-1)^{2m} [1\,(2m)][234 \dots (2m-1)]\,; \quad (1)$$

and afterwards to develop some of the consequences of the formulæ so obtained.

Equation (1) gives the expansion of a Pfaffian of the m-th type in terms of certain Pfaffians of the first type and their complementaries. It would seem probable that similar formulæ should exist for the development of a Pfaffian of the m-th type in terms of Pfaffians of any lower type and their complementaries. We will endeavour to obtain such formulæ by the method of induction.

Now we have

$$\begin{aligned}[123 \dots (2m)] &= -[2134 \dots (2m)] \\ &= -[21][345 \dots (2m)] + [23][145 \dots (2m)] \\ &\quad -[24][135 \dots (2m)] + \dots \\ &\qquad \dots + (-1)^{2m-1}[2\,(2m)][134 \dots (2m-1)] \\ &= [12][345 \dots (2m)] + [23][145 \dots (2m)] \\ &\quad -[24][135 \dots (2m)] + \dots \\ &\qquad \dots + (-1)^{2m-1}[2\,(2m)][134 \dots (2m-1)].\end{aligned}$$

Similarly, we should obtain

$$\begin{aligned}[123 \dots (2m)] &= -[13][245 \dots (2m)] + [23][145 \dots (2m)] \\ &\quad + [34][1256 \dots (2m)] - [35][1246 \dots (2m)] + \dots \\ &\quad \dots + (-1)^{2m-2}[3\,(2m)][1245 \dots (2m-1)].\end{aligned}$$

We have, in all, $2m$ equations of this type, including equation (1).

Adding these together, and dividing the result by 2, we obtain

$$
\begin{aligned}
m[123\ldots(2m)] = \;& [12][345\ldots(2m)]-[13][245\ldots(2m)]+\ldots \\
&\ldots+(-1)^{2m}[1\,(2m)][234\ldots(2m-1)] \\
&+[23][145\ldots(2m)]-[24][135\ldots(2m)] \\
&\ldots+(-1)^{2m-1}[2\,(2m)][134\ldots(2m-1)] \\
&+[34][1256\ldots(2m)]-[35][1246\ldots(2m)]+\ldots \\
&\ldots+(-1)^{2m-2}[3\,(2m)][1245\ldots(2m-1)] \\
&+\ldots.
\end{aligned}
$$

Taking account of equation (1), this reduces to

$$
\begin{aligned}
(m-1)[123\ldots(2m)] = \;& [23][145\ldots(2m)]-[24][135\ldots(2m)]+\ldots \\
&\ldots+(-1)^{2m-1}[2\,(2m)][134\ldots(2m-1)] \\
&+[34][1256\ldots(2m)-[35][1246\ldots(2m)]+\ldots \\
&\ldots+(-1)^{2m-2}[3\,(2m)][1245\ldots(2m-1)]+\ldots.
\end{aligned}
$$

This equation may be written in the form

$$
\begin{aligned}
&(m-1)[123\ldots(2m)] \\
&\quad = \Sigma\pm[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(2m)][pq], \quad (2)
\end{aligned}
$$

where p and q denote two numbers, standing in their natural order, abstracted from the set

$$2,\ 3,\ 4,\ \ldots,\ 2m.$$

The Σ denotes that all possible products of the given type are to be included, the sign of any given product being positive or negative according as the number of displacements required to restore the series

$$p,\ q,\ 2,\ 3,\ 4,\ \ldots,\ (p-1),\ (p+1),\ \ldots,\ (q-1),\ (q+1),\ \ldots,\ (2m)$$

to its natural numerical order is even or odd.

Now the theorem expressed by equation (2) may be applied to each of the coefficients in that equation. Thus we have

$$
\begin{aligned}
&(m-2)[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(2m)] \\
&\quad = \Sigma\pm[12\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots \\
&\qquad\qquad \ldots(s-1)(s+1)\ldots(2m)][rs].
\end{aligned}
$$

Therefore we obtain

$$(m-1)(m-2)[123\ldots(2m)]$$
$$= 2\Sigma\pm[12\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots$$
$$\ldots(s-1)(s+1)\ldots(2m)]$$
$$\times\{[pq][rs]-[pr][qs]+[ps][qr]\},$$

which reduces to

$$\tfrac{1}{2}(m-1)(m-2)[123\ldots(2m)]$$
$$= \Sigma\pm[12\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots$$
$$\ldots(s-1)(s+1)\ldots(2m)][pqrs]. \quad (3)$$

In equation (3), p, q, r, s denote four numbers, standing in their natural order, abstracted from the set

$$2,\ 3,\ 4,\ \ldots,\ 2m.$$

A rule similar to that we gave in the case of equation (2) obtains for the determination of the sign of any particular product included under the Σ.

We may again apply the theorem expressed by equation (2) to the coefficients of the Pfaffians of the type $[pqrs]$ contained in equation (3). Thus suppose p, q, r, s, t, u to be six numbers, standing in their natural numerical order, abstracted from the set

$$2,\ 3,\ 4,\ \ldots,\ 2m.$$

We will denote the Pfaffian formed from the numbers

$$1,\ 2,\ 3,\ 4,\ \ldots,\ 2m$$

by leaving out $\quad p,\ q,\ r,\ s,\ t,\ u$

by the symbol $\quad C\,[pqrstu]$.

Then we have

$$\tfrac{1}{2}(m-1)(m-2)(m-3)[123\ldots(2m)]$$
$$= \Sigma\pm C\,[pqrstu]\,\{\ [pq][rstu]-[pr][qstu]+[ps][qrtu]$$
$$-[pt][qrsu]+[pu][qrst]+[qr][pstu]$$
$$-[qs][prtu]+[qt][prsu]-[qu][prst]$$
$$+[rs][pqtu]-[rt][pqsu]+[ru][pqst]$$
$$+[st][pqru]-[su][pqrt]+[tu][pqrs]\}.$$

But, in virtue of what we have proved above, we see that

$$[pq][rstu]-[pr][qstu]+\ldots+[tu][pqrs] = 3\,[pqrstu].$$

Thus we obtain

$$\frac{(m-1)(m-2)(m-3)}{1.2.3}[123\ldots(2m)] = \Sigma\pm C[pqrstu][pqrstu]. \quad (4)$$

The method of procedure is now evident, as it is easily seen that the same process may be repeated. By its repeated application we arrive at a theorem which may be written in the form

$$\frac{(m-1)(m-2)\ldots(m-n+1)}{(n-1)!}[123\ldots(2m)]$$
$$= \Sigma\pm[1p_1p_2\ldots p_{2n-1}]\,C\,[1p_1p_2\ldots p_{2n-1}]. \quad (5)$$

In this formula $p_1, p_2, \ldots, p_{2n-1}$ denote $2n-1$ numbers, standing in their natural order, abstracted from the set

$$2,\ 3,\ 4,\ \ldots,\ 2m.$$

The sign of any particular product, included under the Σ, is decided by the number of displacements required to restore the series of numbers

$$p_1,\ p_2,\ \ldots,\ p_{2n-1},\ 2,\ 3,\ 4,\ \ldots,\ (p_1-1),\ (p_1+1),\ \ldots$$

to its natural numerical order, according to the customary rule.

2. Reverting to the original definition of a Pfaffian as the square root of a skew-symmetrical determinant of even order, we see that it is easy to justify that the repetition of any one of the numbers occurring within its symbolical expression necessitates the evanescence of the said Pfaffian.

Thus, if in equation (1) we replace $2m$ by unity, we obtain the known theorem

$$[12][1345\ldots(2m-1)]-[13][1245\ldots(2m-1)]+\ldots$$
$$\ldots+(-1)^{2m-1}[1\,(2m-1)][1234\ldots(2m-2)] = 0, \quad (6)$$

which is also capable of an easy direct proof.

In a similar manner we can deduce a new theorem from any special case of the general theorem of the preceding article. Thus we have

$$\Sigma\pm[1pqr][123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots$$
$$\ldots(2m-1)] = 0, \quad (7)$$

and so on.

Now consider the expression

$$[12xy]\,[1345\,\ldots\,(2m-1)\,xy]-[13xy]\,[1245\,\ldots\,(2m-1)\,xy]+\ldots$$
$$\ldots+(-1)^{2m-1}\,[1\,(2m-1)\,xy]\,[1234\,\ldots\,(2m-2)\,xy].$$

On expanding the Pfaffians of the second type we obtain for the coefficient of $[1x]$ in the above expression

$$-\{[2y]\,[1345\,\ldots\,(2m-1)\,xy]-[3y]\,[1245\,\ldots\,(2m-1)\,xy]+\ldots$$
$$\ldots+(-1)^{2m-1}\,[(2m-1)\,y]\,[1234\,(2m-2)\,xy]\}$$
$$=[xy]\,[1234\,\ldots\,(2m-1)\,y]-[1y]\,[2345\,\ldots\,(2m-1)\,xy],$$

by an application of the theorem contained in equation (6).

Hence the above expression becomes

$$[xy]\,\{[12]\,[1345\,\ldots\,(2m-1)\,xy]-[13]\,[1245\,\ldots\,(2m-1)\,xy]+\ldots$$
$$\ldots+(-1)^{2m-1}\,[1\,(2m-1)]\,[1234\,\ldots\,(2m-2)\,xy]$$
$$+[1x]\,[1234\,\ldots\,(2m-1)\,y]-[1y]\,[1234\,\ldots\,(2m-1)\,x]\}=0,$$

by a further application of the theorem contained in equation (6). Thus we obtain

$$[12xy]\,[1345\,\ldots\,(2m-1)\,xy]-[13xy]\,[1245\,\ldots\,(2m-1)\,xy]+\ldots$$
$$\ldots+(-1)^{2m-1}\,[1\,(2m-1)\,xy]\,[1234\,\ldots\,(2m-2)\,xy]=0. \quad (8)$$

Consider next the expression

$$[12x_1y_1x_2y_2]\,[1345\,\ldots\,(2m-1)\,x_1y_1x_2y_2]$$
$$-[13x_1y_1x_2y_2]\,[1245\,\ldots\,(2m-1)\,x_1y_1x_2y_2]+\ldots$$
$$\ldots+(-1)^{2m-1}\,[1\,(2m-1)\,x_1y_1x_2y_2]\,[1234\,\ldots\,(2m-2)\,x_1y_1x_2y_2].$$

Expanding the Pfaffians of the third type, we obtain for the coefficient of $[1x_1]$ in this expression

$$-\{[2y_1x_2y_2]\,[1345\,\ldots\,(2m-1)\,x_1y_1x_2y_2]$$
$$-[3y_1x_2y_2]\,[1245\,\ldots\,(2m-1)\,x_1y_1x_2y_2]+\ldots$$
$$\ldots+(-1)^{2m-1}\,[(2m-1)\,y_1x_2y_2]\,[1234\,\ldots\,(2m-2)\,x_1y_1x_2y_2]\}$$
$$=\quad[x_1y_1x_2y_2]\,[1234\,\ldots\,(2m-1)\,y_1x_2y_2]$$
$$-[1y_1x_2y_2]\,[2345\,\ldots\,(2m-1)\,x_1y_1x_2y_2],$$

by an application of the theorem contained in equation (8). Hence, since we have

$$[1x_1]\,[1y_1x_2y_2]-[1y_1]\,[1x_1x_2y_2]+[1x_2]\,[1x_1y_1y_2]-[1y_2]\,[1x_1y_1x_2]=0,$$

it follows that our expression reduces to

$$\begin{aligned}[x_1y_1x_2y_2]\{\ & [12][1345\ldots(2m-1)\,x_1y_1\,x_2y_2]\\ &-[13][1245\ldots(2m-1)\,x_1y_1\,x_2y_2]+\ldots\\ &\ldots+(-1)^{2m-1}[1\,(2m-1)][1234\ldots(2m-2)\,x_1y_1\,x_2y_2]\\ &+[1x_1][1234\ldots(2m-1)\,y_1\,x_2y_2]\\ &-[1y_1][1234\ldots(2m-1)\,x_1x_2\,y_2]\\ &+[1x_2][1234\ldots(2m-1)\,x_1y_1\,y_2]\\ &-[1y_2][1234\ldots(2m-1)\,x_1y_1\,x_2]\}=0,\end{aligned}$$

by an application of the theorem contained in equation (6). Thus we have

$$\begin{aligned}&[12x_1y_1\,x_2y_2][1345\ldots(2m-1)\,x_1y_1\,x_2y_2]\\ &-[13x_1y_1\,x_2y_2][1245\ldots(2m-1)\,x_1y_1\,x_2y_2]+\ldots\\ &\ldots+(-1)^{2m-1}[1\,(2m-1)\,x_1y_1\,x_2y_2][1234\ldots(2m-2)\,x_1y_1\,x_2y_2]=0. \quad (9)\end{aligned}$$

This process is clearly capable of indefinite repetition. Thus we eventually obtain, as the generalization of the theorems contained in equations (8) and (9), the equation

$$\begin{aligned}&[12x_1y_1\ldots x_ny_n][1345\ldots(2m-1)\,x_1y_1\ldots x_ny_n]\\ &-[13x_1y_1\ldots x_ny_n][1245\ldots(2m-1)\,x_1y_1\ldots x_ny_n]+\ldots\\ &\ldots+(-1)^{2m-1}[1\,(2m-1)\,x_1y_1\ldots x_ny_n][1234\ldots(2m-2)\,x_1y_1\ldots x_ny_n]=0. \quad (10)\end{aligned}$$

We are now in a position to obtain an interesting general theorem which possesses an important application. Thus, consider the expression

$$\begin{aligned}&[12x_1y_1\ldots x_ny_n][345\ldots(2m)\,x_1y_1\ldots x_ny_n]\\ &-[13x_1y_1\ldots x_ny_n][245\ldots(2m)\,x_1y_1\ldots x_ny_n]+\ldots\\ &\ldots+(-1)^{2m}[1\,(2m)\,x_1y_1\ldots x_ny_n][234\ldots(2m-1)\,x_1y_1\ldots x_ny_n].\end{aligned}$$

Expanding the Pfaffians of the $(n+1)$-th type, we obtain for the coefficient of $[1x_1]$ in this expression

$$\begin{aligned}-\{\ &[2y_1\ldots x_ny_n][345\ldots(2m)\,x_1y_1\ldots x_ny_n]\\ &-[3y_1\ldots x_ny_n][245\ldots(2m)\,x_1y_1\ldots x_ny_n]+\ldots\\ &\ldots+(-1)^{2m}[(2m)\,y_1\ldots x_ny_n][234\ldots(2m-1)\,x_1y_1\ldots x_ny_n]\}\\ =-\,&[x_1y_1\ldots x_ny_n][234\ldots(2m)\,y_1\,x_2y_2\ldots x_ny_n],\end{aligned}$$

by an application of the theorem contained in equation (10).* Thus our expression becomes

$$\begin{aligned}[x_1y_1 \dots x_ny_n] \{ \; & [12][345 \dots (2m)\, x_1y_1 \dots x_ny_n] \\ & -[13][245 \dots (2m)\, x_1y_1 \dots x_ny_n] + \dots \\ & \dots + (-1)^{2m} [1\,(2m)][234 \dots (2m-1)\, x_1y_1 \dots x_ny_n] \\ & -[1x_1][234 \dots (2m)\, y_1\, x_2y_2 \dots x_ny_n] \\ & +[1y_1][234 \dots (2m)\, x_1\, x_2y_2 \dots x_ny_n] - \dots \\ & \dots + [1y_n][234 \dots (2m)\, x_1y_1 \dots x_n] \} \\ = [x_1y_1 \dots x_ny_n]&[123 \dots (2m)\, x_1y_1 \dots x_ny_n].\end{aligned}$$

Thus we obtain the general theorem

$$\begin{aligned} & [12x_1y_1 \dots x_ny_n][345 \dots (2m)\, x_1y_1 \dots x_ny_n] \\ & -[13x_1y_1 \dots x_ny_n][245 \dots (2m)\, x_1y_1 \dots x_ny_n] + \dots \\ & \dots + (-1)^{2m} [1\,(2m)\, x_1y_1 \dots x_ny_n][234 \dots (2m-1)\, x_1y_1 \dots x_ny_n] \\ & = [x_1y_1 \dots x_ny_n][123 \dots (2m)\, x_1y_1 \dots x_ny_n]. \qquad (11)\end{aligned}$$

3. It is evident that the process which we have employed in Art. 1 to obtain equations (2), (3), (4), &c., from equation (1) may with equal facility be applied to equation (11). The equation corresponding to (2) is obtained without difficulty. On the left-hand side of the equation corresponding to (3), we shall have

$$(m-1)(m-2)\,[123 \dots (2m)\, x_1y_1 \dots x_ny_n]\, \{[x_1y_1 \dots x_ny_n]\}^2.$$

Corresponding to the expression

$$[pq][rs]-[pr][qs]+[ps][qr]$$

on the right-hand side of the same equation, we shall have

$$\begin{aligned}[pq\, x_1y_1 \dots x_ny_n][rs\, x_1y_1 \dots x_ny_n] - [pr\, x_1y_1 \dots x_ny_n][qs\, x_1y_1 \dots x_ny_n] \\ + [ps\, x_1y_1 \dots x_ny_n][qr\, x_1y_1 \dots x_ny_n],\end{aligned}$$

* To obtain this particular application, write out the theorem on the supposition that the numbers involved are

$$1,\ 2,\ 3,\ \dots,\ (2m+1),\ x_2,\ y_2,\ \dots,\ x_n,\ y_n;$$

and then replace unity by y_1 and $(2m+1)$ by x_1 and transpose the numbers within the square brackets.

which, by means of the theorem contained in equation (11), reduces to

$$[x_1y_1\dots x_ny_n]\,[pqrs\,x_1y_1\dots x_ny_n].$$

Dividing both sides of the equation by $[x_1y_1\dots x_ny_n]$, we obtain the required result.

Similar remarks apply to equation (3). Thus, if we make use of the symbol

$$K\,[1p_1p_2\dots p_{2l-1}]$$

to denote the Pfaffian obtained from the set of numbers

$$1,\ 2,\ 3,\ \dots,\ 2m,\ x_1,\ y_1,\ \dots,\ x_n,\ y_n$$

by leaving out the set $\quad 1,\ p_1,\ p_2,\ \dots,\ p_{2l-1},$

which are all supposed to be abstracted from the set formed by the first $2m$ of these numbers, we arrive at the theorem

$$\Sigma\pm[1p_1p_2\dots p_{2l-1}x_1y_1\dots x_ny_n]\,K\,[1p_1p_2\dots p_{2l-1}]$$
$$=\frac{(m-1)(m-2)\dots(m-l+1)}{(l-1)!}[123\dots(2m)\,x_1y_1\dots x_ny_n]\,[x_1y_1\dots x_ny_n]. \tag{12}$$

It is evident that we may treat this equation in a similar manner to that in which we treated the equations of Art. 1 in order to obtain equations (6) and (7). We shall thus arrive at a general theorem containing that expressed by equation (10) as a particular case. Thus, supposing

$$p_1,\ p_2,\ \dots,\ p_{2l-1}$$

to be any set of numbers abstracted from the set

$$2,\ 3,\ 4,\ \dots,\ 2m-1,$$

and denoting the Pfaffian formed from the set

$$1,\ 2,\ 3,\ 4,\ \dots,\ 2m-1,\ x_1,\ y_1,\ \dots,\ x_n,\ y_n$$

by leaving out these by the symbol

$$K\,[p_1p_2\dots p_{2l-1}],$$

we have $\quad \Sigma\pm[1p_1p_2\dots p_{2l-1}x_1y_1\dots x_ny_n]\,K\,[p_1p_2\dots p_{2l-1}]=0. \qquad (13)$

4. In the theory of the reduction of a Pfaffian expression, we come across Pfaffians of a special form. This special form, though giving rise to certain differential relations of great importance, in no way modifies the form of the algebraic relations by which the Pfaffians are connected. From the theorems developed in the present com-

munication, we can deduce certain relations that are of importance in the said theory. Thus, consider the theorem

$$[12x_1y_1 \ldots x_ny_n][34x_1y_1 \ldots x_ny_n]-[13x_1y_1 \ldots x_ny_n][24x_1y_1 \ldots x_ny_n]$$
$$+[14x_1y_1 \ldots x_ny_n][23x_1y_1 \ldots x_ny_n]$$
$$= [x_1y_1 \ldots x_ny_n][1234x_1y_1 \ldots x_ny_n].$$

As particular cases of this we have a set of theorems of the type

$$[1256 \ldots (2m)][3456 \ldots (2m)]-[1356 \ldots (2m)][2456 \ldots (2m)]$$
$$+[1456 \ldots (2m)][2356 \ldots (2m)] = [56 \ldots (2m)][123 \ldots (2m)], \quad (14)$$

which are of importance in the theory referred to.

On Burmann's Theorem. By A. C. Dixon.

Received September 28th, 1901. Read November 14th, 1901.

The following method gives a proof of Burmann's form of Lagrange's theorem, and also an extension of it which is curious and may possibly be useful.

Let Fx, fx be two functions of the complex variable x, and C a simple contour in the x-plane such that Fx, fx are analytical within and on C, and that $|fx| = k$, a constant, along C.

Let $a_1, a_2, a_3, \ldots$ be the points within C at which fx vanishes, and $b_1, b_2, b_3, \ldots$ those at which fx has a value c such that $|c| < k$. Denote $fx \div (x-a_r)$ by $f_r x$ $(r = 1, 2, \ldots)$, and suppose $f_1a_1, f_2a_2, \ldots$ not to vanish.

The value of
$$\int_{(C)} \frac{Fxf'x\,dx}{fx-c}$$
is
$$2\iota\pi\,\{Fb_1+Fb_2+Fb_3+\ldots\}.$$

But the subject of integration may be expanded in ascending powers of c, since $|c| < |f|$ along C. Hence

$$2\iota\pi\,\{Fb_1+Fb_2+\ldots\}$$
$$= \int_{(C)} \frac{Fxf'x}{fx}\,dx + c\int_{(C)} \frac{Fxf'x}{(fx)^2}\,dx + \ldots + c^n\int_{(C)} \frac{Fxf'x}{(fx)^{n+1}}\,dx + \ldots.$$

The coefficient of c^n may be put in the form

$$\frac{1}{n}\int_{(C)}\frac{F'x}{(fx)^n}\,dx$$

by integrating by parts.

The subject of integration in this coefficient becomes infinite at $a_1, a_2, a_3, \ldots$. Let $C_1, C_2, \ldots$ be small contours described about these points respectively; then

$$\frac{1}{n}\int_{(C_1)}\frac{F'x}{(fx)^n}\,dx = \frac{1}{n}\int_{(C_1)}\frac{F'x}{(x-a_1)^n\,(f_1x)^n}\,dx = \frac{2\iota\pi}{n!}\left(\frac{d}{da_1}\right)^{n-1}\frac{F'a_1}{(f_1a_1)^n}.$$

The coefficient of c^n is therefore

$$\frac{2\iota\pi}{n!}\sum_r\left(\frac{d}{da_r}\right)^{n-1}\frac{F'a_r}{(f_ra_r)^n}.$$

The first term in the expansion is $2\iota\pi\sum_r Fa_r$. Hence

$$\sum_r Fb_r = \sum_r Fa_r + c\sum_r\frac{F'a_r}{f_ra_r} + \ldots + \frac{c^n}{n!}\sum_r\left(\frac{d}{da_r}\right)^{n-1}\frac{F'a_r}{(f_ra_r)^n} + \ldots.$$

Putting 1 for Fx in this result, we find that there are as many points b as points a. If there is only one point a, we have Burmann's theorem. If there are more, the expansion on the right is the sum of a number of series each of the Burmann form; in general these series would not converge separately, but the sum converges absolutely under the conditions stated.

By putting instead of F its square, cube, ..., we may find series for $\sum(Fb_r)^2$, $\sum(Fb_r)^3$, ..., and such expansions might be used to calculate the coefficients in an equation with Fb_1, Fb_2, ... for roots.

The region of validity of the series can be readily assigned. It is bounded by a curve $|fx| = k$, where k is a quantity suitably chosen. Suppose for the moment that Fx has no singularity that affects the question. For small values of k the curve $|fx| = k$ will consist of small ovals enclosing $a_1, a_2, \ldots$ respectively. Within each of these the corresponding Burmann series is valid. As k increases the series still hold good until two of the ovals coalesce—say those about a_1, a_2. There will be an intermediate nodal form of the curve $|fx| = k$. Outside this the Burmann series are not valid singly, but their sum is still a valid representation of the sum of the two values of Fx, until a value of k is reached for which a new oval coalesces with that about a_1, a_2; after this the sum of the two series is not valid, but the sum of three, or possibly more, will still represent the sum

of the corresponding values of Fx. This process may be carried on until the curve $|fx| = k$ reaches a singularity of Fx.

For instance, let $fx \equiv (x-a_1)(x-a_2)(x-a_3)$,

and let x_1, x_2, x_3 be the roots of the equation

$$(x-a_1)(x-a_2)(x-a_3) = c.$$

Then the above method gives expansions for

$$\sin x_1 + \sin x_2 + \sin x_3,$$

$$\cos x_1 + \cos x_2 + \cos x_3,$$

$$e^{kx_1} + e^{kx_2} + e^{kx_3},$$

and so on, in powers of c, and these expansions hold for all finite values of c.

If fx vanishes to a higher order than the first at one of the a points —say a_1—the result must be somewhat modified. We may then put

$$fx = (x-a_1)^{\alpha} f_1 x,$$

where $$f_1 a_1 \neq 0,$$

and the corresponding partial series is the result of putting a_1 for x in

$$\alpha \left[Fx + \frac{c}{\alpha!} \left(\frac{d}{dx}\right)^{\alpha-1} \frac{F'x}{f_1 x} + \ldots + \frac{c^n}{(n\alpha)!} \left(\frac{d}{dx}\right)^{n\alpha-1} \frac{F'x}{(f_1 x)^n} + \ldots \right]:$$

The remainder after n terms is in any case

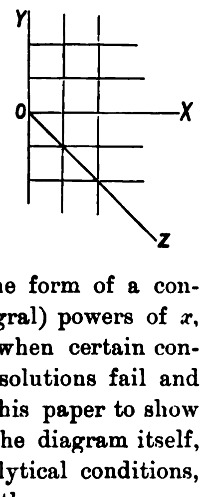
$$\frac{1}{2\iota\pi} c^n \int_{(C)} \frac{Fx f'x}{(fx)^n (fx-c)} dx.$$

There is a similar extension of Taylor's or Maclaurin's theorem, when the function expanded is not uniform, but has a finite number of branches in the region considered.

Suppose, for instance, that fx is a two-valued function within a circle, centre O, radius k, with no singularity within this circle except a branch-point at e. Let $f_1 x$, $f_2 x$ be the two branches of fx. Then Maclaurin's series for either of these holds good within a concentric circle of radius $|e|$. In the ring between the two circles the two series are not available separately, but their sum is still a valid representation of $f_1 x + f_2 x$. Similarly in other cases.

The Puiseux Diagram and Differential Equations. By R. W. H. T. HUDSON. Received October 7th, 1901. Read November 14th, 1901.

The usual method of using a diagram of points, in order to determine the first approximation to a solution of a differential equation corresponding to given initial values, is well known.* It will be convenient in what follows to consider only equations of the first order, but it will be found that equations of higher order can be dealt with in a similar manner. A typical term in the equation is $Cx^a y^b y'^c$, and is represented by the point $(b+c, a-c)$. When the equation is in its complete generality, the diagram consists of all the "unit points" on and between the lines OY, OZ. By a suitable change of variables the point O can be made to disappear, and, when the origin is a *singular point*, other unit points are wanting also; so that an unclosed polygon joins some point on OY to some point on OZ.



As a rule the solution to be obtained is in the form of a converging series of increasing (not necessarily integral) powers of x, but it is known from the analytical theory that, when certain conditions among the coefficients are satisfied, these solutions fail and must be replaced by others. It is the purpose of this paper to show how these cases of failure may be predicted from the diagram itself, and to give an interpretation of the necessary analytical conditions, and to see how far the geometrical character of the curves representing the solutions is affected.

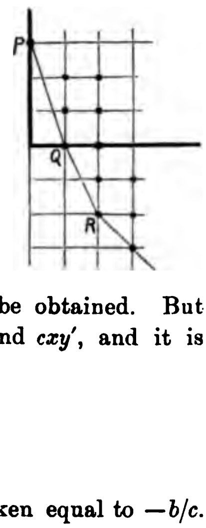
In the application of the diagram to obtain an approximate solution of the form

$$y = Ax^\sigma$$

three matters are to be specially noticed—in general the leading index σ is determined by the *slope* of a side of the polygon; the

* See *Proc. Lond. Math. Soc.*, Vol. XXXIII., p. 392, and the references there given.

leading coefficient A is determined by the *coefficients* of the terms corresponding to points on this side; *one* point in the diagram may correspond to two or more terms in the differential equation. The last consideration is most important and constitutes the main difference between this theory and the corresponding theory of branch-points of algebraic functions. For example, in dealing with an algebraic equation the point (a, b) could arise only from a term Cx^by^n, but in the case of a differential equation this point could correspond also to terms $x^{b+1}y^{n-1}y'$, $x^{b+2}y^{n-2}y'^2$, &c.



Consider, as an example, the equation

$$0 = ax^3+by+cxy'+y'^2+\dots.$$

To the terms written correspond in order the points P, Q, Q, R. The side PQ gives

$$\sigma = 3, \quad a+(b+3c)A = 0;$$

the side QR gives

$$\sigma = 2, \quad b+2c+4A = 0;$$

and so, in general, two regular expansions can be obtained. But here the point Q corresponds to two terms by and cxy', and it is possible to choose σ so that

$$by+cxy' = 0,$$

when

$$y = Ax^\sigma,$$

whatever be the value of A; in fact σ must be taken equal to $-b/c$. Now draw through Q the line whose slope corresponds to this quantity. If $-b/c$ lies between 2 and 3, this line has all the points of the diagram except Q entirely on one side of it, and consequently, by the usual theory of these diagrams, indicates a suitable first approximation. It is to be noticed that the value of σ thus obtained depends on the coefficients in the differential equation, not on the exponents of the different terms, and the coefficient of x^σ is entirely arbitrary. Thus, subject to the inequalities $2 < -b/c < 3$, we obtain a solution in increasing powers of x,

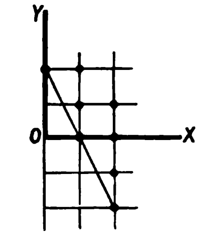
$$y = kx^{-b/c}+\dots,$$

where k is arbitrary.

The theorem of which this is an example will be found in the text-books, where the convergence is rigorously established. The present method is useful as foretelling the possibilities without calculation,

but is, of course, applicable only when the variables are restricted to be real quantities, as is the case in the geometrical theory.

The preceding example is a very simple case, but the same considerations apply to equations of higher order, and when the singularity is more complicated; we have to look out for corners of the polygon arising from more than one term. Points *on* the sides of the polygon also call for attention. Thus in the equation



$$0 = ax^2 + by + cxy' + y'^2 + \dots$$

the only possibility is

$$y = Ax^2 + \dots,$$

where $$0 = a + (b+2c)A + 4A^2.$$

The substitution $$y = Ax^2 + \eta$$

gives a term in the η-equation corresponding to every term in the y-equation except that in x^2; so that the point corresponding to the terms in η and $x\eta'$ is a corner, and this case is reduced to the preceding.

The present example is interesting, because it is of general occurrence in the theory of differential equations of the first order; for the three points $(0, 1)$, $(0, 0)$, $(1, -1)$ which are missing correspond to three conditions which determine a finite number of sets of initial values of x, y, y'. We thus find that when only two inequalities are satisfied, there exist certain points (*nœuds*) on the discriminant locus, at which two special integral curves touch the locus and other integral curves osculate one of the special curves.*

It is sufficient to consider only a side of the polygon which joins points on the first and second columns; for, if the term Ax^σ has been obtained from any other side, the substitution of $Ax^\sigma + y$ instead of y leads to a new diagram, in which the side corresponding to the next term in the solution is of the kind just mentioned. We may suppose σ to be an integer; for, if σ is a fraction with denominator m, we make the substitution $x = \xi^m$, and the leading exponent of ξ becomes integral.

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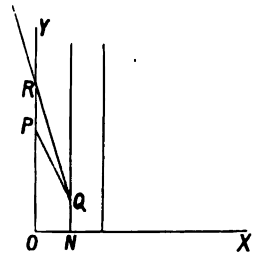
* For drawings of these curves see *Edin. Phil. Trans.*, Vol. XXXVIII., p. 817; *Münchener Sitzungsberichte*, Bd. XXI., p. 23.

It appears from what has been said that the most general case we have to consider is

$$0 = ax^{n+\sigma} + bx^n y + cx^{n+1} y' + \dots .$$



In the diagram

$$OP = n+\sigma, \quad ON = 1, \quad NQ = n.$$

The expansion indicated by PQ is

$$y = Ax^{\sigma} + \dots,$$

where $\quad 0 = a + (b + c\sigma) A.$

The substitution $\quad y = Ax^{\sigma} + \eta$

has the effect of leaving the point Q and of moving P one unit higher, and so successive terms in the expansion can be obtained without fail, unless $-b/c$ is a positive integer equal to or greater than σ. Suppose that $-b/c$ is greater than σ, make $OR = n - b/c$, and join QR; then the slope of QR corresponds to $-b/c$ in the same way as the slope of QP to σ.

If R is not a unit point, the regular solution in integral powers of x can be obtained term by term, but at the stage when P has just crossed R the possibility arises of an expansion beginning with $kx^{-b/c}$, where k is arbitrary; for, at this stage, RQ does not separate the points of the diagram. In this case we have an infinitude of integral curves touching the axis of x at the origin and having contact of a certain order with a special integral curve.

If R falls on a unit point, then when P arrives at R the equation has the form

$$0 = a'x^{n+m} + bx^n y + cx^{n+1} y' + \dots,$$

where $\quad m = -b/c.$

An approximate solution may be found by retaining only the first three terms. Thus:

$$0 = a''x^m - my + xy',$$

$$0 = \frac{a''}{x} + \frac{d}{dx}\left(\frac{y}{x^m}\right),$$

$$y = -a''x^m \log x + kx^m,$$

where k is arbitrary. Thus the solution is no longer regular, but, since $x \log x$ vanishes with x, the terms already obtained up to x^{m-1}

hold good as an approximate solution; so that the geometrical character of the curves in the immediate vicinity of the singular point is not affected by the circumstance that $-\sigma-b/c$ is a positive integer.

We can bring this into agreement with the analytical theory as follows. The equation under consideration is

$$0 = ax^{n+\sigma}+bx^{n}y+cx^{n+1}y'+\dots.$$

Make the substitutions

$$y = (\rho+v)\,x^{\sigma},$$

$$y' = v'x^{\sigma}+(\rho+v)\,\sigma x^{\sigma-1},$$

leading to $\quad 0 = x^{n+\sigma}\left[a+b\,(\rho+v)+c\sigma\,(\rho+v)+cxv'\right]+\dots.$

Choose ρ so that $\qquad 0 = a+b\rho+c\sigma\rho;$

then $\qquad 0 = bv+c\sigma v+cxv'+\dots,$

or $\qquad x\dfrac{dv}{dx} = \lambda v+\mu x+\dots,$

where $\qquad \lambda = -\sigma-b/c.$

According to the general theory, when λ is not a positive integer, but has its real part positive, there is an infinitude of integrals which vanish with x; and they have the form

$$v = \eta+x^{\lambda}\xi,$$

where η is the regular integral, and ξ is a double series of powers of x and x^{λ} acquiring an arbitrary value for $x = 0$. Again, when λ is a positive integer, there exists an infinitude of integrals, vanishing with x, which are regular functions of x and $x\log x$.* The preceding discussion may be regarded as exemplifying these theorems.

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* Forsyth, *Theory of Differential Equations*, Vol. II., pp. 156, 158.

On the Representation of a Group of Finite Order as a Permutation Group, and on the Composition of Permutation Groups. By W. BURNSIDE. Received November 4th, 1901. Read November 14th, 1901.

In writing of linear groups it is becoming almost necessary to have a phrase by which to distinguish substitution groups, in the older and narrower sense in which every operation effects a permutation of the symbols, from groups of linear substitutions in general. In this paper the former will be called *permutation groups.*

Any permutation group with which a given abstract group is either simply or multiply isomorphic will be called a *representation* of that abstract group as a permutation group.

Two such representations of an abstract group in the same number of symbols will be called *equivalent* when the one can be transformed into the other by a linear substitution of non-vanishing determinant. In dealing with the theory of permutation groups by themselves, it is both natural and convenient to consider first the question of equivalence when transformation by permutations only is permitted.

The *mark* of any sub-group of a permutation group is defined as the number of the symbols operated on which are unchanged by every operation of the sub-group. It will be seen that in the theory of the representation of a group as a permutation group, and also in the composition of permutation groups, the marks of the sub-groups in the distinct transitive representations play a part closely analogous to that of the characteristics of the operations in the distinct representations of the group as an irreducible group of linear substitutions.

The main object of this paper is to determine, for any two representations of a group as a permutation group, the question of equivalence or non-equivalence: first, when transformation by permutations only is permitted; and, secondly, for general transformation.

1. Let G be a group of finite order. All the sub-groups of G may be arranged in a series of conjugate sets. The number of these

conjugate sets, including G itself and that constituted by the identical operation alone, will be represented by μ. Let

$$g_1, g_2, \ldots, g_\mu$$

be representatives of the groups in the μ conjugate sets arranged in such a way that the order of g_{s+1} is not less than the order of g_s. The first, g_1, will then be the sub-group consisting of the identical operation alone, and the last g_μ will be G itself. The order of g_s will be denoted by n_s, that of G being n_μ or simply n. Herr Dyck has shown how G may be represented as a transitive permutation group of degree n/n_s in respect of g_s, so that to g_s there corresponds one of the sub-groups of the permutation group which leave a symbol unchanged.

If g'_s belongs to the same conjugate set of sub-groups in G as g_s, the representation of G in respect of g'_s is identical with that in respect of g_s. The μ representations of G in respect of

$$g_1, g_2, \ldots, g_s, \ldots, g_\mu$$

as transitive permutation groups of degrees

$$n, n/n_2, \ldots, n/n_s, \ldots, 1$$

will be denoted by $\quad G_1, G_2, \ldots, G_s, \ldots, G_\mu$.

These include every possible representation of G as a transitive permutation group.

The *mark* of g_s in G_t will be denoted by m_s^t, each of these numbers being by definition either zero or a positive integer. The only sub-groups of G_t whose marks differ from zero are those contained in the sub-groups that leave a symbol unchanged, *i.e.*, in g_t and its conjugates. Hence, if $s \neq t$, and if $n_s \geqslant n_t$, then m_s^t must be zero. Each of the μ symbols m_t^t is greater than zero. Hence in the square array

$$\begin{array}{llll} m_1^1, & m_2^1, & \ldots, & m_\mu^1, \\ m_1^2, & m_2^2, & \ldots, & m_\mu^2, \\ \ldots & \ldots & & \ldots \\ m_1^\mu, & m_2^\mu, & \ldots, & m_\mu^\mu, \end{array}$$

all terms to the right hand of the leading diagonal are zero, and each term in the leading diagonal is different from zero. It follows that the μ sets of marks $\quad m_1^t, m_2^t, \ldots, m_\mu^t, \quad (t = 1, 2, \ldots, \mu),$

are linearly independent in the sense that there can be no system of equations

$$\sum_t a^t m_i^t = 0$$

for each i from 1 to μ.

2. When a permutation group, transitive or intransitive, is transformed by a permutation of the symbols on which it operates, every sub-group must have the same mark in the transformed group as the corresponding sub-group in the original group.* Hence, if we regard two representations of G in the form of permutation groups as equivalent only when one can be transformed into the other by a permutation, the μ representations $G_1, G_2, \ldots, G_\mu$ are distinct. This is, of course, obviously the case for G_s and G_t, when g_s and g_t are not simply isomorphic; but even when g_s and g_t are simply isomorphic, while the numbers of conjugate sub-groups in the s-th and t-th sets is the same, G_s cannot be transformed into G_t by a permutation, so that the set of sub-groups which correspond to g_i in G_s is transformed (for each i) into that which corresponds to g_i in G_t.

incorrect

3. If G is simply or multiply isomorphic with an intransitive permutation group Γ, each transitive constituent of Γ must be equivalent to one of the permutation groups $G_1, G_2, \ldots, G_\mu$. Hence Γ may be represented by the symbol

$$\sum_{i=1}^{i=\mu} a_i G_i,$$

where a_i (zero or a positive integer) denotes the number of transitive constituents of Γ which are equivalent to G_i. Suppose now that G_s and G_t are set up in two distinct sets of symbols—say x's and y's—n/n_s and n/n_t in number. To every operation of G will correspond a permutation of the $n^2/n_s n_t$ products of the x's and the y's. The permutation group that arises when G_s and G_t are thus *compounded* will be represented by $G_s G_t$ (or $G_t G_s$); it will in general be intransitive. The result of thus compounding G_s and G_t may be represented by the symbolical equation

$$G_s G_t = \sum_{i=1}^{i=\mu} k_{sti} G_i, \tag{i}$$

where k_{sti} denotes the number of transitive constituents of the permutation group on the products of the x's and y's which are

* It should be noted at once that this is not necessarily the case if a permutation group is transformed by a linear substitution on the symbols into another permutation group. This point will be considered later and examples will be given.

equivalent to G_i; and there will be an equation of this form for each pair of suffixes s and t. Moreover, these equations admit of a direct arithmetical interpretation. In fact, m_r^s denotes the number of x's which are unchanged by every operation of g_r, and m_r^t the number of similarly unchanged y's. Hence in the permutation group $G_s G_t$ on the products of the x's and the y's just $m_r^s m_r^t$ products are unchanged by every operation of g_r. The relation (i) therefore implies the system of μ arithmetical identities

$$m_r^s m_r^t = \sum_{i=1}^{i=\mu} k_{sti} m_r^i \quad (r = 1, 2, \ldots, \mu). \qquad \text{(ii)}$$

Conversely, since the determinant of the μ sets of marks is not zero, the system of equations last written determine the μ numbers k_{sti} $(i = 1, 2, \ldots, r)$ uniquely. The result of compounding any two of the transitive representations of G is therefore given directly by the complete system of marks.

4. For the tetrahedral group μ is 5, and g_1, g_2, g_3, g_4, g_5 are groups of orders 1, 2, 3, 4, 12. In the corresponding transitive representations G_1, G_2, G_3, G_4, G_5 the marks are given by the table

	g_1	g_2	g_3	g_4	g_5
G_1	12	0	0	0	0
G_2	6	2	0	0	0
G_3	4	0	1	0	0
G_4	3	3	0	3	0
G_5	1	1	1	1	1

In every case it is clear that

$$G_1 G_t = m_1^t G_1$$

and

$$G_\mu G_t = G_t.$$

The remaining relations in this case indicating the composition of the G's are

$$G_2^2 = 2G_1 + 2G_2, \qquad G_3^2 = G_1 + G_3,$$

$$G_2 G_3 = 2G_1, \qquad G_3 G_4 = G_1,$$

$$G_2 G_4 = 3G_2, \qquad G_4^2 = 3G_4.$$

5. Two permutation groups, given by the symbols $\sum_i a_i G_i$ and $\sum_i \beta_i G_i$, can be equivalent (in respect of transformations by permutations) only when they have the same marks for each set of conjugate sub-groups. But, since the sets of marks are independent, the μ equations

$$\sum_i a_i m_r^i = \sum_i \beta_i m_r^i, \quad (i = 1, 2, \ldots, \mu),$$

involve

$$a_i = \beta_i$$

for each i. Hence to each symbol such as $\sum_i a_i G_i$ corresponds a distinct permutation group which represents G. The determination of all the distinct ways in which G may be represented as a permutation group of degree m will therefore be given by the number of distinct solutions of the equation

$$\sum a_i m_1^i = m$$

in positive integers. For example, the number of distinct representations of the tetrahedral group as a permutation group of degree 7 is the number of solutions of

$$12a_1 + 6a_2 + 4a_3 + 3a_4 + a_5 = 7,$$

viz., 6. These six groups are represented by $G_2 + G_5$, $G_3 + G_4$, $G_3 + 3G_5$, $2G_4 + G_5$, $G_4 + 4G_5$, and $7G_5$.

6. When the variables permuted by a permutation group Γ undergo a linear substitution S, the transformed group $S^{-1}\Gamma S$ is not in general a permutation group, though it may be so in particular cases. Thus any permutation group in n symbols is transformed into itself by the substitution

$$\begin{aligned} x_1' &= x_2 + x_3 + \ldots + x_n, \\ x_2' &= x_1 + x_3 + \ldots + x_n, \\ &\ldots \quad \ldots \quad \ldots \\ x_n' &= x_1 + x_2 + \ldots + x_{n-1}. \end{aligned}$$

A further question of the equivalence or non-equivalence of any two representations of G as a permutation group (in the same number of variables) thus arises when, in addition to transformation by permutations, transformations by linear substitutions which preserve the permutation form of the group are admitted.

In this connexion a theorem due to Herr Frobenius* is of great importance. It may be stated as follows:—

Two representations, G' and G'', of a group G of finite order as a group of linear substitutions in the same number of variables can be transformed into each other if, and only if, the sums of the multipliers of the substitutions of G' and G'' which correspond to a given operation S of G are the same for each operation S.

In any representation of G as a permutation group the sum of the multipliers of the permutation that corresponds to S is the mark of the cyclical sub-group generated by this permutation; for the sum of the multipliers that arise from any cyclical component of the permutation is zero. Hence it follows at once from Herr Frobenius's theorem that the necessary and sufficient conditions that $\sum_i \alpha_i G_i$ and $\sum_i \beta_i G_i$ should be equivalent when transformations by linear substitutions are admitted is that the equation

$$\sum_i (\alpha_i - \beta_i)\, m_t^i = 0$$

should be satisfied by the marks of each cyclical sub-group. Unless G is a cyclical group, the system of equations that thus arise when $\alpha_i - \beta_i$ $(i = 1, 2, \ldots, \mu)$ are regarded as unknown must have (integral) solutions other than all zero values; since their number cannot exceed $\mu - 1$. Hence there must always be representations of a non-cyclical group as a permutation group which, while not equivalent in respect of transformation by permutations, are, in fact, equivalent for transformation by linear substitutions.

7. Suppose that G has s distinct conjugate sets of cyclical subgroups, and (slightly modifying the previous notation) let $g_1, g_2, \ldots, g_s$ be representatives of them. The equations

$$\Sigma\, \alpha_i m_t^i = 0, \quad (t = 1, 2, \ldots, s),$$

will have just $\mu - s$ linearly independent solutions in positive or negative integers, since the determinant

$$\| m_t^i \| \quad (t, i = 1, 2, \ldots, s)$$

* "Ueber die Darstellung der endlichen Gruppen durch lineare Substitutionen," *Berliner Sitzungsberichte*, 1897, pp. 1000–1005.

is different from zero. Moreover,* there is a set of solutions

$$a_1^r,\ a_2^r,\ \ldots,\ a_\mu^r,\quad (r = 1, 2, \ldots, \mu-s),$$

in terms of which the general solution takes the form

$$a_i = \sum_r k_r a_i^r,$$

where the k's are $\mu-s$ arbitrary integers, positive or negative. Hence, if $\Sigma\, a_i G_i$ and $\Sigma\, \beta_i G_i$ are equivalent representations of G in respect of transformation by linear substitutions, then there must be a set of integers k such that

$$a_i - \beta_i = \sum_r k_r a_i^r.$$

Every possible equivalence of the kind considered will therefore arise from the $\mu-s$ fundamental equivalences denoted symbolically by

$$\Sigma\, a_i^r G_i = 0.$$

This is to be understood in the sense that, after removing to the right-hand side the terms with negative coefficients, the permutation groups denoted by the symbols on either side of the equation are equivalent.

8. In illustration, the tetrahedral group may be again considered. The cyclical sub-groups are those denoted by g_1, g_2, g_3 in the preceding table. The three equations among the a's are

$$\begin{aligned} 12a_1 + 6a_2 + 4a_3 + 3a_4 + a_5 &= 0, \\ 2a_2 \qquad + 3a_4 + a_5 &= 0, \\ a_3 \qquad + a_5 &= 0. \end{aligned}$$

The fundamental solutions of these are

$$a_1 = 0,\quad a_2 = -1,\quad a_3 = 1,\quad a_4 = 1,\quad a_5 = -1;$$

and $$a_1 = 1,\quad a_2 = -3,\quad a_3 = 0,\quad a_4 = 2,\quad a_5 = 0.$$

The equivalences corresponding to these, from which all others arise by addition, are

$$G_3 + G_4 = G_2 + G_5,$$

and $$G_1 + 2G_4 = 3G_2.$$

It will be perhaps not without interest to verify directly the equivalences thus indicated. The tetrahedral group is defined abstractly

* Elliott, *Algebra of Quantics*, p. 192.

by the relations $$A^2 = 1, \quad B^3 = 1, \quad (AB)^3 = 1.$$

In the form $G_2 + G_3$ it is represented as a permutation group of degree 7 in which one symbol x_0 is unaltered by every operation, and the others $x_1, x_2, \ldots, x_6$ are permuted transitively. We may take

$$A = (x_1 x_4)(x_2 x_5), \quad B = (x_1 x_2 x_3)(x_4 x_5 x_6).$$

Consider now the seven linear functions of the x's

$$\begin{aligned}
y_1 &= x_0 + x_1 + x_2 + x_3, \\
y_2 &= x_0 + x_3 + x_4 + x_5, \\
y_3 &= x_0 + x_1 + x_5 + x_6, \\
y_4 &= x_0 + x_2 + x_4 + x_6, \\
z_1 &= x_1 + x_4, \\
z_2 &= x_2 + x_5, \\
z_3 &= x_3 + x_6.
\end{aligned}$$

They are obviously linearly independent, and therefore these seven equations give a linear substitution S. Moreover, when the x's undergo the permutations A and B, the y's and z's undergo the permutations
$$(y_1 y_2)(y_3 y_4)$$
and
$$(y_2 y_3 y_4)(z_1 z_2 z_3).$$

Hence the y's and z's are each permuted among themselves by every operation of the group, and the permutation group so arrived at is $G_3 + G_4$; or
$$S\,(G_2 + G_3)\,S^{-1} = G_3 + G_4.$$

Again, in the form $G_1 + 2G_4$ the tetrahedral group is represented as a permutation group of degree 18, interchanging 12 x's, 3 y's, and 3 z's among themselves. The permutations given by A and B may be taken as
$$(x_1 x_2)(x_3 x_4)(x_5 x_6)(x_7 x_8)(x_9 x_{10})(x_{11} x_{12})$$
and
$$(x_1 x_5 x_9)(x_2 x_7 x_{11})(x_3 x_6 x_{12})(x_4 x_8 x_{10})(y_1 y_2 y_3)(z_1 z_2 z_3).$$

The 18 linear functions

$$\begin{aligned}
u_1 &= y_1 + x_1 + x_2, & v_1 &= z_1 + x_1 + x_3, & w_1 &= x_1 + x_4, \\
u_2 &= y_2 + x_5 + x_7, & v_2 &= z_2 + x_5 + x_6, & w_2 &= x_5 + x_8, \\
u_3 &= y_3 + x_9 + x_{11}, & v_3 &= z_3 + x_9 + x_{12}, & w_3 &= x_9 + x_{10}, \\
u_4 &= y_1 + x_3 + x_4, & v_4 &= z_1 + x_2 + x_4, & w_4 &= x_2 + x_3, \\
u_5 &= y_2 + x_6 + x_8, & v_5 &= z_2 + x_7 + x_8, & w_5 &= x_6 + x_7, \\
u_6 &= y_3 + x_{10} + x_{12}, & v_6 &= z_3 + x_{10} + x_{11}, & w_6 &= x_{11} + x_{12}
\end{aligned}$$

are linearly independent and give a linear substitution T. Moreover, the operations A and B give the permutations

$$(u_2u_5)(u_3u_6)(v_1v_4)(v_3v_6)(w_1w_4)(w_2w_5)$$

and $$(u_1u_2u_3)(u_4u_5u_6)(v_1v_2v_3)(v_4v_5v_6)(w_1w_2w_3)(w_4w_5w_6).$$

The u's, v's, and w's therefore undergo a permutation group denoted by $3G_2$ and
$$T(G_1+2G_4)\,T^{-1}=3G_2.$$

9. It may happen that among the $\mu-s$ fundamental equivalences
$$\Sigma a_i^r G_i = 0$$
one or more of the form $$G_i - G_j = 0$$
occur; indicating that two distinct transitive representations of G can be transformed the one into the other by a suitably chosen linear substitution. The necessary and sufficient conditions for this are that the marks of each cyclical sub-group shall be the same in G_i and G_j. Hence g_i and g_j must be of the same order, the conjugate sets to which g_i and g_j belong must contain the same number of conjugate sub-groups, and any cyclical sub-group which enters in a number of sub-groups of the one set must enter in an equal number of the other set.

As an example, these conditions are satisfied by the two distinct sets of octahedral sub-groups which enter in the simple group G of order 168. In respect of an octahedral sub-group g of the one set, G is represented as the transitive group of degree 7 generated by

$$S=(x_1x_2x_3x_6x_4x_5x_7),\quad T=(x_2x_3x_4)(x_5x_6x_7),\quad U=(x_2x_7x_6x_3)(x_4x_5);$$

and an octahedral group g' of the second set is generated by T and V, where $$V=(x_1x_7x_5x_6)(x_2x_3).$$

The seven linear functions

$$y_1=x_2+x_3+x_4,\quad y_2=x_3+x_5+x_6,\quad y_3=x_4+x_6+x_7,$$
$$y_4=x_2+x_5+x_7,\quad y_5=x_1+x_3+x_7,\quad y_6=x_1+x_4+x_5,$$
$$y_7=x_1+x_2+x_6$$

are independent; so that these equations give a linear substitution. When G is transformed by this substitution, it remains a permuta-

tion group, and

$$S = (y_1y_2y_3y_6y_4y_5y_7),\quad T = (y_2y_3y_4)(y_5y_6y_7),\quad U = (y_1y_4y_3y_2)(y_5y_7),$$
$$V = (y_2y_7y_5y_4)(y_3y_6).$$

Hence in the transformed group g' is the sub-group which leaves one symbol y_1 unchanged, while g permutes the symbols in the two transitive sets y_1, y_2, y_3, y_4 and y_5, y_6, y_7. The given linear substitution therefore transforms the transitive representation of G in respect of g into that in respect of g'.

Linear Null Systems of Binary Forms. By J. H. Grace.

Read November 14th, 1901. Received November 18th, 1901.

As an exercise on Hilbert's paper "Ueber vollen Invariantensysteme," *Math. Ann.*, Vol. XLI., I propose to investigate the necessary and sufficient condition that all the combinants of three binary forms which are pure invariants should vanish. The method used applies equally well to any number of binary forms.

Suppose the three forms are

$$f = a_0x_1^n + na_1x_1^{n-1}x_2 + \ldots + a_nx_2^n = a_x^n,$$
$$\phi = b_0x_1^n + nb_1x_1^{n-1}x_2 + \ldots + b_nx_2^n = b_x^n,$$
$$\psi = c_0x_1^n + nc_1x_1^{n-1}x_2 + \ldots + c_nx_2^n = c_x^n;$$

then the combinants in question are such mutual invariants of f, ϕ, ψ as remain unaltered when any of the forms is replaced by a linear combination $lf + m\phi + n\psi$.

The combinants are well known to be rational integral functions of the determinants of the type

$$\begin{vmatrix} a_\alpha & a_\beta & a_\gamma \\ b_\alpha & b_\beta & b_\gamma \\ c_\alpha & c_\beta & c_\gamma \end{vmatrix},$$

whether they involve the variables or not. We shall denote the above determinant by $p_{\alpha\beta\gamma}$.

2. The most important combinant is the covariant

$$J = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2}, & \dfrac{\partial^2 f}{\partial x_1 \partial x_2}, & \dfrac{\partial^2 f}{\partial x_2^2} \\ \dfrac{\partial^2 \phi}{\partial x_1^2}, & \dfrac{\partial^2 \phi}{\partial x_1 \partial x_2}, & \dfrac{\partial^2 \phi}{\partial x_2^2} \\ \dfrac{\partial^2 \psi}{\partial x_1^2}, & \dfrac{\partial^2 \psi}{\partial x_1 \partial x_2}, & \dfrac{\partial^2 \psi}{\partial x_2^2} \end{vmatrix}$$

or
$$(bc)(ca)(ab)\, a_x^{n-2} b_x^{n-2} c_x^{n-2}.$$

It is of order $3n-6$, the coefficient of x_1^{3n-6} is p_{012}, and generally the coefficient of $x_1^{3n-6-\rho} x_2^{\rho}$ is a linear combination of such p's as satisfy the condition
$$\alpha+\beta+\gamma = \rho+3.$$

Now every invariant of J is manifestly a combinant of f, ϕ, ψ, and hence a necessary condition is that all the invariants of J should vanish. But Hilbert has shown that, if all the invariants of a binary form of order m vanish, then the form has a factor of multiplicity equal to the integer next greater than $\dfrac{m}{2}$; hence in our case J has a linear factor of multiplicity greater than $\frac{1}{2}(3n-6)$. Without loss of generality we may suppose this factor to be x_2.

3. Now consider how many times a factor occurs in J when its mode of occurrence in f, ϕ, ψ is given. For perfect generality suppose the factor (x_2) occurs ν times in every form of the system
$$lf+m\phi+n\psi;$$
then there will be two linearly independent forms, say f_1 and f_2, each containing the factor more than ν times—say μ times. Finally, there will be a single form included in
$$l_1 f_1 + l_2 f_2,$$
containing the given factor more than μ times—say λ times. Hence, given the factor, we can choose three forms included in the system—say f', ϕ', ψ'—such that f' contains the factor λ times, ϕ' contains the factor μ times, and ψ' contains the factor ν times $(\lambda > \mu > \nu)$; and, of course, the integers λ, μ, ν are quite determined by the linear factor under consideration.

We shall clearly not lose generality (since the combinants are the same for any three members of the system) if we suppose f, ϕ, ψ to

be the same as f', ϕ', ψ' respectively, and further, since we are dealing with invariant properties, we may suppose the given factor to be x_2.

Hence, referring to the definitions of f, ϕ, ψ, we have

$$\left.\begin{array}{l} c_0,\ c_1,\ \ldots,\ c_{\nu-1} = 0, \quad c_\nu \neq 0 \\ b_0,\ b_1,\ \ldots,\ b_{\mu-1} = 0, \quad b_\mu \neq 0 \\ a_0,\ a_1,\ \ldots,\ a_{\lambda-1} = 0, \quad a_\lambda \neq 0 \end{array}\right\} \quad (\lambda > \mu > \nu).$$

It follows easily that, if $\alpha+\beta+\gamma < \lambda+\mu+\nu$, then

$$p_{\alpha\beta\gamma} = 0.$$

For $(\alpha-\lambda)+(\beta-\mu)+(\gamma-\nu) < 0$, and we may suppose that $\alpha > \beta > \gamma$; therefore either $\alpha < \lambda$ or $\beta < \mu$ or $\gamma < \nu$.

If $\alpha < \lambda$, then $\beta < \lambda$ and $\gamma < \lambda$; therefore

$$a_\alpha = a_\beta = a_\gamma = 0;$$

therefore $$p_{\alpha\beta\gamma} = 0.$$

If $\beta < \mu$, then $\gamma < \mu$, $\beta < \lambda$, $\gamma < \lambda$; therefore

$$a_\beta = a_\gamma = b_\beta = b_\gamma = 0;$$

therefore $$p_{\alpha\beta\gamma} = 0.$$

If $\gamma < \nu$, then $\gamma < \mu$, $\gamma < \lambda$; therefore

$$a_\gamma = b_\gamma = c_\gamma = 0;$$

therefore $$p_{\alpha\beta\gamma} = 0.$$

Again, if $$\alpha+\beta+\gamma = \lambda+\mu+\nu,$$

then $$p_{\alpha\beta\gamma} = 0,$$

unless $$\alpha = \lambda, \quad \beta = \mu, \quad \gamma = \nu,$$

for otherwise $\alpha < \lambda$ or $\beta < \mu$ or $\gamma < \nu$.

As regards $p_{\lambda\mu\nu}$, it is equal to $a_\lambda a_\mu a_\nu$, and is therefore not zero.

But in the expression for J the coefficient of $x_1^{3n-6-\rho} x_2^\rho$ is a linear combination of p's for which

$$\alpha+\beta+\gamma = \rho+3,$$

so that, if $\rho+3 < \lambda+\mu+\nu$, the coefficient of $x_1^{3n-6-\rho} x_2^\rho$ is zero, but, if

$$\rho+3 = \lambda+\mu+\nu,$$

it is certainly not zero; that is, the factor x_2 occurs precisely $(\lambda+\mu+\nu-3)$ times in J.

4. We have seen, however, that the factor x_2 must occur more than $\frac{1}{2}(3n-6)$ times in J, and accordingly we have

$$\lambda+\mu+\nu-3 > \tfrac{1}{2}(3n-6);$$

therefore $$\lambda+\mu+\nu > \frac{3n}{2}.$$

Now $$p_{\alpha\beta\gamma} = 0,$$

if $$\alpha+\beta+\gamma < \lambda+\mu+\nu;$$

therefore when $$p_{\alpha\beta\gamma} \neq 0$$

we must have $$\alpha+\beta+\gamma > \lambda+\mu+\nu > \frac{3n}{2}.$$

5. The weight of a product of powers of a's, b's, and c's is the sum of the suffixes of the various letters in the product. If the weight of an invariant of partial degrees i_1, i_2, i_3 of three forms of orders n_1, n_2, n_3 be w, then we have

$$2w = n_1 i_1 + n_2 i_2 + n_3 i_3.$$

In our case $$n_1 = n_2 = n_3, \quad i_1 = i_2 = i_3 = i,$$

where i is the degree of the invariant in the p's; for every p is linear in the coefficients of each form.

Hence for a combinant we must have

$$2w = 3ni.$$

Now the weight of $p_{\alpha\beta\gamma}$ is $\alpha+\beta+\gamma$, and for a non-vanishing p we must have

$$\alpha+\beta+\gamma > \frac{3n}{2},$$

and hence for a product of p's of degree i which does not vanish we must have

$$w > \frac{3ni}{2},$$

and, inasmuch as for terms of a combinant

$$w = \frac{3ni}{2},$$

it follows immediately that all the combinants vanish.

Hence the necessary and sufficient condition that all the combinants should vanish is that J should have a factor of multiplicity greater than $\frac{1}{2}(3n-6)$.

In accordance with the general reasoning of Hilbert, we infer that all combinants of three binary forms are integral algebraic functions of invariants of J, and therefore, *a fortiori*, of the coefficients of J.

The results for any number of binary forms are exactly the same.

Addition Theorems for Hyperelliptic Integrals. *By* A. L. Dixon.

Received and read November 14th, 1901.

The present communication is a continuation of my paper on "An Addition Theorem for Hyperelliptic Theta-Functions," presented to the Society in December, 1900 (*Proc. Lond. Math. Soc.*, Vol. XXXIII., No. 755).

The method there given of deducing theorems in the theory of hyperelliptic integrals from the geometrical properties of confocals is applied to the investigation of addition theorems for the integrals of the second and third kinds.*

I must record my obligation to a paper by Herr O. Staude, on the "Geometrische Deutung der Additionstheoreme der hyperelliptischen Integrale" (*Math. Ann.*, Bd. XXII., 1883). In particular the fundamental idea of § 4 has been taken from that paper.

References to my first paper are prefixed by the number I.

Integrals of the Second Kind.

1. Taking the equations (11), I., § 2, of the straight lines through the point h_ι, which lie in the surfaces S and T, one of them is given by

$$\frac{\xi_p}{\sqrt{p-s\,.\,p-t\,.\,q-r}} = \frac{\xi_q}{\sqrt{q-s\,.\,q-t\,.\,r-p}} = \frac{\xi_r}{\sqrt{r-s\,.\,r-t\,.\,p-q}},$$

$$\xi_s = 0, \quad \xi_t = 0.$$

Let S be the distance measured along this line from h_ι. Then

$$S = \sqrt{\Sigma\,(x_\iota - h_\iota)^2} = \sqrt{\xi_p^2+\xi_q^2+\xi_r^2}\,; \tag{1}$$

and therefore

$$\frac{\iota S}{\sqrt{q-r\,.\,r-p\,.\,p-q}} = \frac{\xi_p}{\sqrt{p-s\,.\,p-t\,.\,q-r}} = \dots. \tag{2}$$

* A paper on the application of the method to confocal conicoids in ordinary space and the deduction of theorems for elliptic integrals has appeared in the *Quarterly Journal*, No. 131, 1902.

Therefore also

$$\frac{2\iota\, dS}{\sqrt{q-r\,.\,r-p\,.\,p-q}} = \frac{2 ds_p}{\sqrt{p-s\,.\,p-t\,.\,q-r}} = \dots \tag{3}$$

$$= \frac{\sqrt{p-q\,.\,p-r}}{\sqrt{q-r}}\,\frac{dp}{\sqrt{P}} = \dots, \tag{4}$$

$$2dS = (p-q)(p-r)\,\frac{dp}{\sqrt{P}} = (q-r)(q-p)\,\frac{dq}{\sqrt{Q}} = (r-p)(r-q)\,\frac{dr}{\sqrt{R}};$$

and therefore
$$2dS = p^2\,\frac{dp}{\sqrt{P}} + q^2\,\frac{dq}{\sqrt{Q}} + r^2\,\frac{dr}{\sqrt{R}}. \tag{5}$$

Integrating, we get

$$2S = \int p^2\,\frac{dp}{\sqrt{P}} - \int p_0^2\,\frac{dp_0}{\sqrt{P_0}} + \int q^2\,\frac{dq}{\sqrt{Q}} - \int q_0^2\,\frac{dq_0}{\sqrt{Q_0}} + \int r^2\,\frac{dr}{\sqrt{R}} - \int r_0^2\,\frac{dr_0}{\sqrt{R_0}},$$

where
$$S = \sqrt{\Sigma\,(x_\iota - h_\iota)^2} = \sqrt{\frac{p_0-q_0\,.\,p_0-r_0}{p_0-s\,.\,p_0-t}}\;\xi_p = \dots.$$

2. Since
$$\Sigma x_\iota^2 = \Sigma a_\iota + p + q + r + s + t,$$
$$\Sigma h_\iota^2 = \Sigma a_\iota + p_0 + q_0 + r_0 + s + t,$$

we get

$$S^2 = 2\Sigma a_\iota + 2s + 2t + p + p_0 + q + q_0 + r + r_0$$
$$-2\Sigma_\iota \frac{(a_\iota+s)(a_\iota+t)}{f'(-a_\iota)}\sqrt{a_\iota+p\,.\,a_\iota+p_0\,.\,a_\iota+q\,.\,a_\iota+q_0\,.\,a_\iota+r\,.\,a_\iota+r_0}, \tag{7}$$

where s and t may be given any value we please, and, in fact, the coefficients of $s+t$ and st vanish by I. (15).

Putting $s = -a_4$, $t = -a_5$, I get

$$S^2 = 2\,(a_1+a_2+a_3) + p + p_0 + q + q_0 + r + r_0$$
$$-2\,\frac{\sqrt{a_1+p\,.\,a_1+p_0\,.\,a_1+q\,.\,a_1+q_0\,.\,a_1+r\,.\,a_1+r_0}}{(a_1-a_2)(a_1-a_3)}$$
$$-2\,\frac{\sqrt{a_2+p\,.\,a_2+p_0\,.\,a_2+q\,.\,a_2+q_0\,.\,a_2+r\,.\,a_2+r_0}}{(a_2-a_3)(a_2-a_1)}$$
$$-2\,\frac{\sqrt{a_3+p\,.\,a_3+p_0\,.\,a_3+q\,.\,a_3+q_0\,.\,a_3+r\,.\,a_3+r_0}}{(a_3-a_1)(a_3-a_2)} \tag{8}$$

$$\text{Also}\quad \xi_p = \frac{\Sigma_\iota \dfrac{h_\iota x_\iota}{a_\iota+p_0} - 1}{\left(\Sigma \dfrac{h_\iota^2}{(a_\iota+p_0)^2}\right)^{\frac{1}{2}}} = \frac{(s-p_0)(t-p_0)\,\Sigma \dfrac{h_\iota x_\iota}{(a_\iota+p_0)(a_\iota+s)(a_\iota+t)}}{\left(\Sigma \dfrac{h_\iota^2}{(a_\iota+p_0)^2}\right)^{\frac{1}{2}}},$$

since $\xi_r = 0, \quad \xi_s = 0$;

and therefore

$$S = \sqrt{P_0}\left\{\Sigma_\iota \frac{\sqrt{a_\iota+p\,.\,a_\iota+q\,.\,a_\iota+q_0\,.\,a_\iota+r\,.\,a_\iota+r_0}}{f'(-a_\iota)\sqrt{a_\iota+p_0}}\right\}. \tag{9}$$

3. Another expression for S, which will be used hereafter, is obtained as follows. We have

$$S = \frac{\sqrt{p_0-q_0\,.\,p_0-r_0}}{\sqrt{p_0-s\,.\,p_0-t}}\,\xi_p = \frac{\sqrt{P_0}}{(p_0-s)(p_0-t)}\left(\Sigma \frac{h_\iota x_\iota}{a_\iota+p_0} - 1\right), \tag{10}$$

where s and t are arbitrary constants.

Putting $s = -a_1$, $t = -a_2$, I get

$$\frac{\sqrt{a_1+p_0\,.\,a_2+p_0}}{\sqrt{a_3+p_0\,.\,a_4+p_0\,.\,a_5+p_0}}\,S = \Sigma \frac{\sqrt{a_3+p\,.\,a_3+p_0\,.\,a_3+q\,.\,a_3+q_0\,.\,a_3+r\,.\,a_3+r_0}}{(a_3-a_4)(a_3-a_5)(a_3+p_0)} - 1, \tag{11}$$

the other two terms in the Σ corresponding to a_4 and a_5. Also, interchanging p and p_0, q and q_0, r and r_0, I get

$$\frac{\sqrt{a_1+p\,.\,a_2+p}}{\sqrt{a_3+p\,.\,a_4+p\,.\,a_5+p}}\,(-S) = \Sigma \frac{\sqrt{a_3+p\,.\,a_3+p_0\,.\,a_3+q\,.\,a_3+q_0\,.\,a_3+r\,.\,a_3+r_0}}{(a_3-a_4)(a_3-a_5)(a_3+p)} - 1.$$

Then, by subtraction,

$$\frac{S}{p-p_0}\left\{\frac{\sqrt{a_1+p_0\,.\,a_2+p_0}}{\sqrt{a_3+p_0\,.\,a_4+p_0\,.\,a_5+p_0}} + \frac{\sqrt{a_1+p\,.\,a_2+p}}{\sqrt{a_3+p\,.\,a_4+p\,.\,a_5+p}}\right\}$$
$$= \frac{\sqrt{a_3+q\,.\,a_3+q_0\,.\,a_3+r\,.\,a_3+r_0}}{(a_3-a_4)(a_3-a_5)\sqrt{a_3+p\,.\,a_3+p_0}} + \frac{\sqrt{a_4+q\,.\,a_4+q_0\,.\,a_4+r\,.\,a_4+r_0}}{(a_4-a_5)(a_4-a_3)\sqrt{a_4+p\,.\,a_4+p_0}}$$
$$+ \frac{\sqrt{a_5+q\,.\,a_5+q_0\,.\,a_5+r\,.\,a_5+r_0}}{(a_5-a_3)(a_5-a_4)\sqrt{a_5+p\,.\,a_5+p_0}}. \tag{12}$$

Integrals of the Third Kind.

4. To find corresponding expressions applicable to integrals of the third kind, let us take the generalized conception of distance as given by Cayley in his sixth memoir upon quantics (*Coll. Works*, Vol. II., pp. 583–592).

Taking for the absolute the continuum

$$\Sigma_\iota \frac{x_\iota^2}{a_\iota+n} = 1^* \quad (\iota = 1, 2, 3, 4, 5),$$

the distance S' between any two points x_ι and h_ι is given by

$$\cos S' = \frac{\Sigma \frac{x_\iota h_\iota}{a_\iota+n} - 1}{\left(\Sigma \frac{h_\iota^2}{a_\iota+n} - 1\right)^{\frac{1}{2}} \left(\Sigma \frac{x_\iota^2}{a_\iota+n} - 1\right)^{\frac{1}{2}}}; \tag{13}$$

and therefore

$$\sin^2 S' = \frac{\left(\Sigma \frac{h_\iota^2}{a_\iota+n} - 1\right)\left(\Sigma \frac{x_\iota^2}{a_\iota+n} - 1\right) - \left(\Sigma \frac{h_\iota x_\iota}{a_\iota+n} - 1\right)^2}{\left(\Sigma \frac{h_\iota^2}{a_\iota+n} - 1\right)\left(\Sigma \frac{x_\iota^2}{a_\iota+n} - 1\right)}$$

$$= \frac{-\Pi\,(a_\iota+n)}{(n-p)(n-q)(n-r)(n-s)(n-t)}$$

$$\times \left\{ \frac{\xi_p^2}{n-p_0} + \frac{\xi_q^2}{n-q_0} + \frac{\xi_r^2}{n-r_0} + \frac{\xi_s^2}{n-s} + \frac{\xi_t^2}{n-t} \right\}. \dagger \tag{14}$$

5. To find an expression for dS' at any point, suppose the point h_1 to move up to and ultimately coincide with the point x_ι, and we get, writing

$$N \equiv \Pi\,(a_\iota+n), \quad N' \equiv \Pi\,(n-\lambda) \quad (\lambda = p, q, r, s, t),$$

$$dS'^2 = -\frac{N}{N'} \left\{ \frac{ds_p^2}{n-p} + \frac{ds_q^2}{n-q} + \frac{ds_r^2}{n-r} + \frac{ds_s^2}{n-s} + \frac{ds_t^2}{n-t} \right\} \dagger \tag{15}$$

$$= -\tfrac{1}{4}\frac{N}{N'} \Sigma \frac{(p-q)(p-r)(p-s)(p-t)}{(n-p)\,P}\, dp^2. \tag{16}$$

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* Staude, *loc. cit.*, *Math. Ann.*, Bd. XXII., p. 23, § 7.

† ξ_p, ξ_q, d_{sp}, d_{sq}, ... have exactly the same meaning here as in the last section, that is, they represent the same expressions in x_ι, or in p, q, r, s, t.

Now along the straight lines considered, namely those which lie in both the surfaces S and T,

$$ds = 0, \quad dt = 0,$$

and, as in the preceding section, we get

$$\frac{(n-p)(n-q)(n-r)\,2dS'}{\sqrt{N\,.\,q-r\,.\,r-p\,.\,p-q}} = \frac{2ds_n}{\sqrt{p-s\,.\,p-t\,.\,q-r}} = \dots \tag{17}$$

$$= \frac{\sqrt{p-q\,.\,p-r}}{\sqrt{q-r}}\,\frac{dp}{\sqrt{P}} = \dots, \tag{18}$$

$$\frac{(n-p)(n-q)(n-r)}{\sqrt{N}}\,2\iota dS' = (p-q)(p-r)\frac{dp}{\sqrt{P}} = (q-r)(q-p)\frac{dq}{\sqrt{Q}}$$

$$= (r-p)(r-q)\frac{dr}{\sqrt{R}},$$

$$\frac{2\iota dS'}{\sqrt{N}} = \frac{dp}{(n-p)\sqrt{P}} + \frac{dq}{(n-q)\sqrt{Q}} + \frac{dr}{(n-r)\sqrt{R}}. \tag{19}$$

Integrating, we get

$$2\iota S' = \int\frac{\sqrt{N}\,dp}{(n-p)\sqrt{P}} - \int\frac{\sqrt{N}\,dp_0}{(n-p_0)\sqrt{P_0}} + \int\frac{\sqrt{N}\,dq}{(n-q)\sqrt{Q}} - \int\frac{\sqrt{N}\,dq_0}{(n-q_0)\sqrt{Q_0}}$$

$$+ \int\frac{\sqrt{N}\,dr}{(n-r)\sqrt{R}} - \int\frac{\sqrt{N}\,dr_0}{(n-r_0)\sqrt{R_0}}. \tag{20}$$

6. One expression for S' is given by

$$\cos S' = \frac{\Sigma\,\dfrac{x_\iota h_\iota}{a_\iota+n} - 1}{\left(\Sigma\,\dfrac{h_\iota^2}{a_\iota+n}-1\right)^{\frac{1}{2}}\left(\Sigma\,\dfrac{x_\iota^2}{a_\iota+n}-1\right)^{\frac{1}{2}}}.$$

$$= \frac{N\left\{\Sigma_\iota\,\dfrac{(a_\iota+s)(a_\iota+t)}{(a_\iota+n)f'(-a_\iota)}\sqrt{a_\iota+p\,.\,a_\iota+p_0\,.\,a_\iota+q\,.\,a_\iota+q_0\,.\,a_\iota+r\,.\,a_\iota+r_0}-1\right\}}{(n-s)(n-t)\sqrt{n-p\,.\,n-p_0\,.\,n-q\,.\,n-q_0\,.\,n-r\,.\,n-r_0}} \tag{21}$$

where s and t may have any value. Putting $s = \infty$, $t = -a_5$, I get

$$\cos S' = \frac{(a_1+n)(a_2+n)(a_3+n)(a_4+n)}{\sqrt{n-p\,.\,n-p_0\,.\,n-q\,.\,n-q_0\,.\,n-r\,.\,n-r_0}}$$

$$\times\left\{\Sigma\,\frac{\sqrt{a_1+p\,.\,a_1+p_0\,.\,a_1+q\,.\,a_1+q_0\,.\,a_1+r\,.\,a_1+r_0}}{(a_1+n)(a_1-a_2)(a_1-a_3)(a_1-a_4)}\right\}, \tag{22}$$

the other three terms of the Σ corresponding to a_2, a_3, and a_4. Also, putting $s = t = \infty$, I get another form, viz.,

$$\cos S' = \frac{N\Sigma_\iota \dfrac{\sqrt{a_\iota+p\,.\,a_\iota+p_0\,.\,a_\iota+q\,.\,a_\iota+q_0\,.\,a_\iota+r\,.\,a_\iota+r_0}}{(a_\iota+n)\,f'(-a_\iota)}}{\sqrt{n-p\,.\,n-p_0\,.\,n-q\,.\,n-q_0\,.\,n-r\,.\,n-r_0}}. \tag{23}$$

Another expression for S' follows from (14) and (2). For, putting

$$\frac{\xi_p}{\sqrt{p_0-s\,.\,p_0-t\,.\,q_0-r_0}} = \frac{\xi_q}{\sqrt{q_0-s\,.\,q_0-t\,.\,r_0-p_0}}$$

$$= \frac{\xi_r}{\sqrt{r_0-s\,.\,r_0-t\,.\,p_0-q_0}} = \frac{\iota S}{\sqrt{q_0-r_0\,.\,r_0-p_0\,.\,p_0-q_0}}$$

in (14), we get

$$\sin^2 S' = -\frac{NS^2}{(n-p)(n-q)(n-r)(n-p_0)(n-q_0)(n-r_0)},$$

or

$$\sin S' = \frac{\iota\sqrt{NS}}{\sqrt{n-p\,.\,n-p_0\,.\,n-q\,.\,n-q_0\,.\,n-r\,.\,n-r_0}}. \tag{24}$$

Confocals of Revolution.

7. It is also interesting from the geometrical point of view to consider the results obtained when two of the parameters a are equal to one another, and one of the families degenerates into the system of planes through an axis.

It will be found that in this way a real geometrical construction is obtained for the sum of integrals of the third kind.

Take
$$\Sigma \frac{x_\iota^2}{a_\iota+\lambda} = 1 \quad (\iota = 1, 2, 3, 4, 5, 6),$$

where
$$x_1^2 = y^2+z^2,$$

so that y, z, x_2, x_3, x_4, x_5, x_6 are Cartesian coordinates, as the equation of a set of confocal ${}_2R_6$'s of revolution in a space S_7, and let q, r, s, t, u, v be the values of λ for the six members of the set through any point. The degenerate seventh member of the set corresponding to the parameter p is given by

$$y = z\tan\theta.$$

Then, as before,

$$\Sigma \frac{x_\iota^2}{a_\iota+\lambda}-1 \equiv -\frac{(\lambda-q)(\lambda-r)(\lambda-s)(\lambda-t)(\lambda-u)(\lambda-v)}{\Pi\,(a_\iota+\lambda)};$$

$$x_\iota^2 = -\frac{(a_\iota+q)(a_\iota+r)(a_\iota+s)(a_\iota+t)(a_\iota+u)(a_\iota+v)}{f'(-a_\iota)},$$

writing $$f(\lambda) \equiv \Pi\,(a_\iota+\lambda);$$

and $4ds_q^2 = \Sigma_\iota \dfrac{x_\iota^2}{(a_\iota+q)^2}\,dq^2 = \dfrac{(q-r)(q-s)(q-t)(q-u)(q-v)}{f(q)}\,dq^2, \ldots,$

but ds_p^2 is replaced by $x_1^2 d\theta^2$, that is, by

$$-\frac{(a_1+q)(a_1+r)(a_1+s)(a_1+t)(a_1+u)(a_2+v)}{f'(-a_1)}\,d\theta^2.$$

8. In considering the "tangent cone," we may without loss of generality take the coordinates of the point h_ι to be $0, h_1, h_2, h_3, h_4, h_5, h_6$; so that its equation is

$$\left(\frac{y^2+z^2}{a_1+\lambda}+\Sigma_\iota\frac{x_\iota^2}{a_\iota+\lambda}-1\right)\left(\frac{h_1^2}{a_1+\lambda}+\Sigma_\iota\frac{h_\iota^2}{a_\iota+\lambda}-1\right)$$

$$=\left(\frac{zh_1}{a_1+\lambda}+\Sigma_\iota\frac{x_\iota h_\iota}{a_\iota+\lambda}-1\right)^2 \quad (\iota = 2, 3, 4, 5, 6),$$

which when referred to its principal axes takes the form

$$\frac{y^2}{a_1+\lambda}+\frac{\xi_q^2}{\lambda-q}+\frac{\xi_r^2}{\lambda-r}+\frac{\xi_s^2}{\lambda-s}+\frac{\xi_t^2}{\lambda-t}+\frac{\xi_u^2}{\lambda-u}+\frac{\xi_v^2}{\lambda-v}=0.$$

Then, exactly as before (I., § 12), the common points of the three surfaces T, U, V and the three tangent planes T', U', V' are given by

$$\xi_t = 0, \quad \xi_u = 0, \quad \xi_v = 0,$$

$$\frac{y^2}{(a_1+t)(a_1+u)(a_1+v)(q-r)(r-s)(s-q)}$$

$$=\frac{-\xi_q^2}{(q-t)(q-u)(q-v)(r-s)(a_1+s)(a_1+r)}$$

$$=\frac{\xi_r^2}{(r-t)(r-u)(r-v)(a_1+s)(a_1+q)(q-s)}$$

$$=\frac{-\xi_s^2}{(s-t)(s-u)(s-v)(a_1+q)(q-r)(a_1+r)}, \qquad \text{(A)}$$

putting $-a_1$ for p in the equations of I., § 12.

Now, writing, for y, $x_1 d\theta$; for ξ_q, ds_q; &c., we get, as the differential equations of the eight lines in the surfaces T, U, V,

$$\frac{1}{(q-r)(r-s)(s-q)}\,\frac{2\,d\theta}{\sqrt{f'(-a_1)}} = \frac{\epsilon}{(r-s)(a_1+r)(a_1+s)}\,\frac{dq}{\sqrt{a_1+q}\,\sqrt{f(q)}}$$

$$= \frac{\epsilon'}{(s-q)(a_1+s)(a_1+q)}\,\frac{dr}{\sqrt{a_1+r}\,\sqrt{f(r)}}$$

$$= \frac{\epsilon''}{(q-r)(a_1+q)(a_1+r)}\,\frac{ds}{\sqrt{a_1+s}\,\sqrt{f(s)}}.$$

Writing $\quad Q \equiv (a_2+q)(a_3+q)(a_4+q)(a_5+q)(a_6+q), \; \ldots,$

these are equivalent to

$$\epsilon\,\frac{dq}{\sqrt{Q}} + \epsilon'\,\frac{dr}{\sqrt{R}} + \epsilon''\,\frac{ds}{\sqrt{S}} = 0,$$

$$\epsilon\,\frac{q\,dq}{\sqrt{Q}} + \epsilon'\,\frac{r\,dr}{\sqrt{R}} + \epsilon''\,\frac{s\,ds}{\sqrt{S}} = 0,$$

$$\frac{2\,d\theta}{\sqrt{f'(-a_1)}} + \frac{\epsilon\,dq}{(a_1+q)\sqrt{Q}} + \frac{\epsilon'\,dr}{(a_1+r)\sqrt{R}} + \frac{\epsilon''\,ds}{(a_1+s)\sqrt{S}} = 0,$$

and the integral of these is

$$\cos\theta\,\frac{(a_1+\lambda)(a_1+\mu)\sqrt{a_1+q\,.\,a_1+q_0\,.\,a_1+r\,.\,a_1+r_0\,.\,a_1+s\,.\,a_1+s_0}}{f'(-a_1)}$$

$$+\Sigma_\iota\,\frac{(a_\iota+\lambda)(a_\iota+\mu)\sqrt{a_\iota+q\,.\,a_\iota+q_0\,.\,a_\iota+r\,.\,a_\iota+r_0\,.\,a_\iota+s\,.\,a_\iota+s_0}}{f'(-a_\iota)} + 1 = 0$$

$$(\iota = 2, 3, 4, 5, 6),$$

which is the same as (21).

9. Also putting $\qquad S^2 = y^2+\xi_q^2+\xi_r^2+\xi_s^2,$

we get, from (A),

$$\frac{S^2}{(a_1+q_0)(a_1+r_0)(a_1+s_0)} = \frac{y^2}{(a_1+t)(a_1+u)(a_1+v)};$$

and therefore, substituting for y,

$$S = \frac{\sqrt{a_1+q\,.\,a_1+r\,.\,a_1+s\,.\,a_1+q_0\,.\,a_1+r_0\,.\,a_1+s_0}}{\sqrt{a_2-a_1\,.\,a_3-a_1\,.\,a_4-a_1\,.\,a_5-a_1\,.\,a_6-a_1}}\sin\theta,$$

which is equation (24).

In fact θ represents the "distance" between two points, when the absolute is taken to be

$$y^2+z^2 = 0.$$

Paraboloids.

10. The degenerate case of paraboloids may also be briefly noticed. The particular case here worked out gives an integral of the two equations

$$\sum_{r=1}^{r=6} u_r = 0, \quad \sum_{r=1}^{r=6} \Pi(u_r a) = 0,$$

where u is an ordinary elliptic integral of the first kind, and $\Pi(u, a)$ one of the third kind. Starting with

$$\Sigma \frac{x_\iota^2}{a_\iota+\lambda} = 4a(x+a\lambda) \quad (\iota = 1, 2, 3, 4),$$

I shall get

$$\Sigma \frac{x_\iota^2}{a_\iota+\lambda} - 4a(x+a\lambda) = -\frac{4a^2(\lambda-p)(\lambda-q)(\lambda-r)(\lambda-s)(\lambda-t)}{\prod_\iota (a_\iota+\lambda)}.$$

Then $$x_\iota^2 = \frac{4a^2(a_\iota+p)(a_\iota+q)(a_\iota+r)(a_\iota+s)(a_\iota+t)}{f'(-a_\iota)},$$

where $$f(\lambda) = \Pi(a_\iota+\lambda).$$

Also $$-\frac{x}{a} = p+q+r+s+t,$$

and so

$$4ds_p^2 = dp^2\left(\Sigma \frac{x_\iota^2}{(a_\iota+p)^2}+4a^2\right) = \frac{4a^2(p-q)(p-r)(p-s)(p-t)}{f(p)}\,dp^2.$$

The results of I., §§ 2, 3 will not be altered, and I shall get an algebraical integral of the equations

$$\int\frac{dp}{\sqrt{P}} - \int\frac{dp_0}{\sqrt{P_0}} + \epsilon\left(\int\frac{dq}{\sqrt{Q}} - \int\frac{dq_0}{\sqrt{Q_0}}\right) + \epsilon'\left(\int\frac{dr}{\sqrt{R}} - \int\frac{dr_0}{\sqrt{R_0}}\right) = 0,$$

$$\int\frac{p\,dp}{\sqrt{P}} - \int\frac{p_0\,dp_0}{\sqrt{P_0}} + \epsilon\left(\int\frac{q\,dq}{\sqrt{Q}} - \int\frac{q_0\,dq_0}{\sqrt{Q_0}}\right) + \epsilon'\left(\int\frac{r\,dr}{\sqrt{R}} - \int\frac{r_0\,dr_0}{\sqrt{R_0}}\right) = 0,$$

where $$\Theta \equiv f(\theta) \equiv (a_1+\theta)(a_2+\theta)(a_3+\theta)(a_4+\theta),$$

in the form

$$\Sigma\frac{(a_\iota+\lambda)\sqrt{\overline{a_\iota+p\,.\,a_\iota+p_0\,.\,a_\iota+q\,.\,a_\iota+q_0\,.\,a_\iota+r\,.\,a_\iota+r_0}}}{f'(-a_\iota)}$$
$$+p+p_0+q+q_0+r+r_0+2\lambda = 0 \quad (\iota = 1, 2, 3, 4),$$

where λ is an arbitrary constant.

11. I proceed to express the results obtained in the notation adopted in the former communication (I., §§ 5, 6). Making the substitutions

$$p = \frac{1}{a-x}, \quad a_1 = \frac{1}{b-a}, \quad a_2 = \frac{1}{c-a}, \quad \ldots,$$

and writing $\Theta \equiv a-\theta . b-\theta . c-\theta . d-\theta . e-\theta . f-\theta,$

I take

$$\left. \begin{aligned} u &= \int \frac{(e-x)\,dx}{(e-f)\sqrt{X}} - \int \frac{(e-y)\,dy}{(e-f)\sqrt{Y}} \\ v &= \int \frac{(f-x)\,dx}{(f-e)\sqrt{X}} - \int \frac{(f-y)\,dy}{(f-e)\sqrt{Y}} \end{aligned} \right\}, \tag{25}$$

from which

$$\left. \begin{aligned} (x-y)\frac{dx}{\sqrt{X}} &= (f-y)\,du + (e-y)\,dv \\ (x-y)\frac{dy}{\sqrt{Y}} &= (f-x)\,du + (e-x)\,dv \end{aligned} \right\}. \tag{26}$$

Then $\int \frac{p^2 dp}{\sqrt{P}} - \int \frac{p_0^2 dp_0}{\sqrt{P_0}}$ becomes

$$\sqrt{b-a . c-a . d-a . e-a . f-a}\left\{\int \frac{dx}{(a-x)\sqrt{X}} - \int \frac{dy}{(a-y)\sqrt{Y}}\right\},$$

and

$$\int \frac{dx}{(a-x)\sqrt{X}} - \int \frac{dy}{(a-y)\sqrt{Y}} = \int \frac{a+f-x-y}{(a-x)(a-y)}\,du + \int \frac{a+e-x-y}{(a-x)(a-y)}\,dv.$$

But

$$F^2 = \zeta^2 (f-x)(f-y),$$

$$A^2 = \alpha^2 (a-x)(a-y);$$

therefore

$$F^2/\zeta^2 - A^2/\alpha^2 = (f-a)(a+f-x-y),$$

and so

$$\int \frac{dx}{(a-x)\sqrt{X}} - \int \frac{dy}{a-y\sqrt{Y}} = \frac{u}{a-f} + \frac{v}{a-e} + \int \frac{\alpha^2 F^2 du}{(f-a)\,\zeta^2 A^2} + \int \frac{\alpha^2 E^2 dv}{(e-a)\,\epsilon^2 A^2},$$

and therefore $\int \frac{p^2 dp}{\sqrt{P}} - \int \frac{p_0^2 dp_0}{\sqrt{P_0}}$ becomes

$$\sqrt{b-a . c-a . d-a . e-a . f-a}$$
$$\times \left\{\frac{u}{a-f} + \frac{v}{a-e} + \int \frac{\alpha^2 F^2 du}{(f-a)\,\zeta^2 A^2} + \int \frac{\alpha^2 E^2 dv}{(e-a)\,\epsilon^2 A^2}\right\}$$

Now $b-a$, $a-f$, ... can be expressed in terms of $\alpha, \beta, \gamma, \ldots$,* and I finally get equation (6) in the form

$$\frac{2S}{(\alpha\beta\gamma\delta\epsilon\zeta)^{\frac{1}{2}}} = \Sigma_r \int \frac{aF_r^2\,du_r}{(\overline{a\zeta})\,\zeta A_r^2} + \frac{aE_r^2\,dv_r}{(\overline{a\epsilon})\,\epsilon A_r^2}, \qquad (27)$$

where, as in I., § 9, $(\overline{a\zeta})$ is written as an abbreviation for

$$(a\beta\zeta)(a\gamma\zeta)(a\delta\zeta)(a\epsilon\zeta).$$

Then, using $Z_A(u, v)$ to denote the function

$$\int \frac{1}{A^2}\left(\frac{aF^2du}{(\overline{a\zeta})\,\zeta} + \frac{aE^2dv}{(\overline{a\epsilon})\,\epsilon}\right),$$

I have
$$\frac{2S}{(\alpha\beta\gamma\delta\epsilon\zeta)^{\frac{1}{2}}} = Z_A(u_1, v_1) + Z_A(u_2, v_2) + Z_A(u_3, v_3), \qquad (28)$$

where
$$u_1+u_2+u_3 = 0 \quad \text{and} \quad v_1+v_2+v_3 = 0.$$

12. Let us now consider the transformation of S^2 as given in § 2 (8). We may obviously put

$$S^2 = \lambda_0 + \lambda\Sigma\frac{B_r^2}{A_r^2} + \mu\Sigma\frac{C_r^2}{A_r^2} + \nu\Sigma\frac{D_r^2}{A_r^2} + \lambda'\frac{B_1B_2B_3}{A_1A_2A_3} + \mu'\frac{C_1C_2C_3}{A_1A_2A_3} + \nu'\frac{D_1D_2D_3}{A_1A_2A_3},$$

where $\lambda_0, \lambda, \lambda', \ldots$ are coefficients to be determined. Then, firstly, since there is a linear relation between the squares of A, B, C, D, λ_0 may be merged in λ, μ, ν; and, secondly, since S^2 must vanish when

$$u_3, v_3 = 0, 0 \quad \text{and} \quad u_1, v_1 = -u_2, -v_2,$$

we have
$$\beta\lambda' + 2\lambda a = 0, \ \ldots,$$

and we may therefore put

$$\begin{aligned} S^2 = \ & \lambda\left(\frac{B_1^2}{A_1^2} + \frac{B_2^2}{A_2^2} + \frac{B_3^2}{A_3^2} - 2\frac{a}{\beta}\,\frac{B_1B_2B_3}{A_1A_2A_3}\right) \\ & + \mu\left(\frac{C_1^2}{A_1^2} + \frac{C_2^2}{A_2^2} + \frac{C_3^2}{A_3^2} - 2\frac{a}{\gamma}\,\frac{C_1C_2C_3}{A_1A_2A_3}\right) \\ & + \nu\left(\frac{D_1^2}{A_1^2} + \frac{D_2^2}{A_2^2} + \frac{D_3^2}{A_3^2} - 2\frac{a}{\delta}\,\frac{D_1D_2D_3}{A_1A_2A_3}\right). \end{aligned}$$

---

* For the values of a, $(a\beta\zeta)$, ... in terms of a, b, c, d, e, f, see Cayley, *Coll. Works*, Vol. x., pp. 502, 503. Neglecting a fourth root of unity which occurs as a coefficient, it is easily found that

$$a-b = (\overline{a\beta})/a\beta, \ \ldots,$$

and that
$$\sqrt{b-a\,.\,c-a\,.\,d-a\,.\,e-a\,.\,f-a} = (a\beta\gamma\delta\epsilon\zeta)^{\frac{1}{2}}/a^2.$$

But the coefficient of $-2\frac{B_1B_2B_3}{A_1A_2A_3}$ in S^2 as determined from (8) is $\frac{(c-a)(d-a)}{(b-a)(b-c)(b-d)}\frac{\alpha^3}{\beta^3}$, which, on substituting $c-a=(\overline{\alpha\gamma})/\alpha\gamma$, becomes on reduction $\frac{\alpha^2}{(\alpha\epsilon\zeta)^2}\frac{(\beta\epsilon\zeta)^2}{(\overline{\alpha\beta})}$, and so we get, finally,

$$
\begin{aligned}
S^2 = {} & \frac{\alpha\beta\,(\beta\epsilon\zeta)^2}{(\overline{\alpha\beta})(\alpha\epsilon\zeta)^2}\left(\frac{B_1^2}{A_1^2}+\frac{B_2^2}{A_2^2}+\frac{B_3^2}{A_3^2}-2\frac{\alpha}{\beta}\frac{B_1B_2B_3}{A_1A_2A_3}\right) \\
& + \frac{\alpha\gamma\,(\gamma\epsilon\zeta)^2}{(\overline{\alpha\gamma})(\alpha\epsilon\zeta)^2}\left(\frac{C_1^2}{A_1^2}+\frac{C_2^2}{A_2^2}+\frac{C_3^2}{A_3^2}-2\frac{\alpha}{\gamma}\frac{C_1C_2C_3}{A_1A_2A_3}\right) \\
& + \frac{\alpha\delta\,(\delta\epsilon\zeta)^2}{(\overline{\alpha\delta})(\alpha\epsilon\zeta)^2}\left(\frac{D_1^2}{A_1^2}+\frac{D_2^2}{A_2^2}+\frac{D_3^2}{A_3^2}-2\frac{\alpha}{\delta}\frac{D_1D_2D_3}{A_1A_2A_3}\right), \qquad (29)
\end{aligned}
$$

where $$\frac{2S}{(\alpha\beta\gamma\delta\epsilon\zeta)^4}=Z_A(u_1,v_1)+Z_A(u_2,v_2)+Z_A(u_3,v_3),$$

which is one form of the addition theorem for integrals of the second kind.

13. The transformation of formula (12), § 3, leads similarly to

$$S\frac{(AEF)_1}{(\alpha\epsilon\zeta)}\frac{\alpha^2}{(\alpha\beta\gamma\delta\epsilon\zeta)^4}\frac{A_1^2}{\alpha^2}\frac{\beta\gamma\delta}{B_1C_1D_1}=\Sigma\frac{\beta^2\gamma\delta}{(\overline{\beta\gamma})(\overline{\beta\delta})}\frac{\alpha B_2B_3A_1}{\beta A_2A_3B_1};$$

and therefore

$$
\begin{aligned}
S = {} & \frac{(\alpha\beta\gamma\delta\epsilon\zeta)^4}{(\overline{\beta\gamma})(\overline{\beta\delta})(\overline{\gamma\delta})}\frac{\alpha\,(\alpha\epsilon\zeta)\,B_1C_1D_1}{(AEF)_1A_1A_2A_3} \\
& \times\left\{(\overline{\gamma\delta})\frac{B_2B_3}{B_1}+(\overline{\beta\delta})\frac{C_2C_3}{C_1}-(\overline{\beta\gamma})\frac{D_2D_3}{D_1}\right\}. \qquad (30)
\end{aligned}
$$

The signs of the terms in the bracket are determined by putting

$$(u_1,v_1)=(ef),\quad (u_2,v_2)=-(u_3,v_3)-(ef),$$

when S is seen to vanish by the help of the identical relation

$$(\gamma\delta\epsilon)(\gamma\delta\zeta)(BEF)\,B+(\beta\delta\epsilon)(\beta\delta\zeta)(CEF)\,C-(\beta\gamma\epsilon)(\beta\gamma\zeta)(DEF)\,D=0.$$

This gives the result that, with

$$\Sigma_r u_r=0,\quad \Sigma_r v_r=0\quad (r=1,2,3),$$

$$
\begin{aligned}
\Sigma_r Z_A(u_r,v_r) = {} & \frac{2\alpha\,(\alpha\epsilon\zeta)\,B_1C_1D_1}{(AEF)_1A_1A_2A_3} \\
& \times\left\{\frac{B_2B_3}{(\overline{\beta\gamma})(\overline{\beta\delta})\,B_1}+\frac{C_2C_3}{(\overline{\beta\gamma})(\overline{\gamma\delta})\,C_1}-\frac{D_2D_3}{(\overline{\beta\delta})(\overline{\gamma\delta})\,D_1}\right\}. \qquad (31)
\end{aligned}
$$

14. In exactly the same way $\int \frac{\sqrt{N}\,dp}{(n-p)\sqrt{P}}$ (§ 5) is replaced by $\int \frac{\sqrt{N}\,dx}{(n-x)\sqrt{X}}$, and $\int \frac{\sqrt{N}\,dx}{(n-x)\sqrt{X}} - \frac{\sqrt{N}\,dy}{(n-y)\sqrt{Y}}$ is equal to

$$\int \frac{\sqrt{N}\,F^2 du}{(f-n)\,\zeta^2\,(n-x)(n-y)} + \int \frac{\sqrt{N}\,E^2 dv}{(e-n)\,\epsilon^2\,(n-x)(n-y)}.$$

Now take
$$\left.\begin{array}{l} B(\alpha,\beta) = B' = \beta\sqrt{b-n\,.\,b-a} \\ C(\alpha,\beta) = C' = \gamma\sqrt{c-n\,.\,c-a} \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\}, \tag{32}$$

so that α, β are parameters of double Θ-functions for which $A(\alpha,\beta)$ vanishes identically. Then, since

$$\begin{aligned} &\frac{(n-x)(n-y)}{(b-n)(c-n)(d-n)} \\ &= \frac{(b-x)(b-y)}{(b-n)(b-c)(b-d)} + \ldots \\ &= \frac{\beta\gamma\delta}{\alpha\,(\overline{\alpha\epsilon})(\overline{\alpha\zeta})}\left\{(\alpha\beta\epsilon)^2(\alpha\beta\zeta)^2\frac{B^2}{B'^2} - (\alpha\gamma\epsilon)^2(\alpha\gamma\zeta)^2\frac{C^2}{C'^2} + (\alpha\delta\epsilon)^2(\alpha\delta\zeta)^2\frac{D^2}{D'^2}\right\}, \end{aligned}$$

where the signs are determined from the identity

$$(\alpha\beta\epsilon)^2(\alpha\beta\zeta)^2 - (\alpha\gamma\epsilon)^2(\alpha\gamma\zeta)^2 + (\alpha\delta\epsilon)^2(\alpha\delta\zeta)^2 = 0,$$

I get
$$\int \frac{\sqrt{N}}{(n-x)(n-y)}\left(\frac{F^2 du}{(f-n)\,\zeta^2} + \frac{E^2 dv}{(e-n)\,\epsilon^2}\right)$$

$$\begin{aligned} &= \int \frac{\alpha\beta\gamma\delta\,(\alpha\epsilon\zeta)}{(\overline{\alpha\epsilon})(\overline{\alpha\zeta})}\,\frac{B'C'D'}{(AEF)'} \\ &\times \frac{(\overline{\alpha\zeta})\,E^2F^2du + (\overline{\alpha\epsilon})\,\zeta F^2E^2dv}{(\alpha\beta\epsilon)^2(\alpha\beta\zeta)^2B^2C'^2D'^2 - (\alpha\gamma\epsilon)^2(\alpha\gamma\zeta)^2C^2D'^2B'^2 + (\alpha\delta\epsilon)^2(\alpha\delta\zeta)^2D^2B'^2C'^2}. \end{aligned} \tag{33}$$

This I shall denote by $2\Pi(u, v;\, n)$, or by $2\Pi(u, v;\, \alpha, \beta)$, and for shortness I write

$$\begin{aligned} M^2 &\equiv (\alpha\beta\epsilon)^2(\alpha\beta\zeta)^2B^2C'^2D'^2 - (\alpha\gamma\epsilon)^2(\alpha\gamma\zeta)^2C^2D'^2B'^2 + (\alpha\delta\epsilon)^2(\alpha\delta\zeta)^2D^2B'^2C'^2 \\ &\equiv \frac{(\alpha\beta\gamma\delta\epsilon\zeta)^2}{\alpha^2}(n-x)(n-y). \end{aligned} \tag{34}$$

Then formula (20) becomes

$$\iota S' = \Pi(u_1, v_1;\, n) + \Pi(u_2, v_2;\, n) + \Pi(u_3, v_3;\, n). \tag{35}$$

The transformation of (22) gives

$$\cos S' = \frac{(\overline{\alpha\zeta})}{\alpha\beta\gamma\delta\epsilon\zeta} \frac{B'^2 C'^2 D'^2 E'^2}{M_1 M_2 M_3}$$
$$\times \left\{ \frac{(\overline{\alpha\beta})^2 (\overline{\beta\zeta}) B_1 B_2 B_3}{B'^2} + \frac{(\overline{\alpha\gamma})^2 (\overline{\gamma\zeta}) C_1 C_2 C_3}{C'^2} + \frac{(\overline{\alpha\delta})^2 (\overline{\delta\zeta}) D_1 D_2 D_3}{D'^2} \right.$$
$$\left. - \frac{(\overline{\alpha\epsilon})^2 (\overline{\epsilon\zeta}) E_1 E_2 E_3}{E'^2} \right\}, \quad (36)$$

and, of (24),

$$\sin S' = \frac{\iota S \alpha^2 \beta\gamma\delta\epsilon\zeta\, (\alpha\epsilon\zeta)\, A_1 A_2 A_3 B' C' D' E'^2 F'^2}{(\overline{\alpha\epsilon})(\alpha\zeta)\, M_1 M_2 M_3\, (AEF)'}, \quad (37)$$

which may be written

$$\Sigma \Pi\, (u_r, v_r;\ \alpha, \beta)$$
$$= \sinh^{-1} \frac{\alpha\, (\alpha\beta\gamma\delta\epsilon\zeta)^{\frac{1}{2}}\, (\alpha\epsilon\zeta)\, A_1 A_2 A_3 B' C' D' E'^2 F'^2\, \Sigma Z_A (u_r, v_r)}{2\, (\overline{\alpha\epsilon})(\overline{\alpha\zeta})\, M_1 M_2 M_3\, (AEF)'},$$

when $\Sigma u_r = 0, \quad \Sigma v_r = 0 \quad (r = 1, 2, 3).$ (38)

Linear Groups in an Infinite Field. By L. E. Dickson, Ph.D.

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1. *Introduction.*

Various branches of analytic group theory may be coordinated and generalized by the study of groups of transformations in an arbitrary field or domain of rationality. A *field* (Körper) is a set of elements within which the rational operations of algebra may be performed. Thus the totality of rational numbers forms a field R; the totality of all complex numbers $a+b\sqrt{-1}$ forms a field C. A finite field is completely defined by its order, which is necessarily a power of a prime number p, the latter being the *modulus* of the field. Although certain infinite fields may have a modulus p, so that $\mu+p\equiv\mu$, $\tau p\equiv 0$, for arbitrary elements μ, τ in the field, such fields do not seem to have been investigated. An example is given by the aggregate of the Galois fields of orders p^n, for $n = 1, 2, 3, \ldots$.

In a memoir entitled "Theory of Linear Groups in an Arbitrary Field," to appear in the *Transactions of the American Mathematical Society*, October, 1901, I exhibit four infinite systems of groups of transformations which are simple groups in every field. For the case of the field C of all complex numbers, they are the simple continuous groups of Lie, shown by Killing and Cartan to give the only systems of simple continuous groups of a finite number of parameters. Among them is the linear homogeneous group defined by the quadratic invariant*

$$\xi_1\eta_1+\xi_2\eta_2+\ldots+\xi_m\eta_m.$$

The structure of this group, for the case of a finite field, was determined by the writer in the *Proc. Lond. Math. Soc.*, Vol. XXX., pp. 70–98; with slight modifications, the investigation there made holds for any infinite field. Another group investigated in the *Trans. Amer. Math. Soc.*, is that defined by the invariant

$$\xi_0^2+\xi_1\eta_1+\xi_2\eta_2+\ldots+\xi_m\eta_m.$$

The next problem in the same direction is the study of the group Γ_m of all linear homogeneous transformations in an arbitrary field F which leave absolutely invariant

$$\xi_0^2-\nu\eta_0^2+\xi_1\eta_1+\xi_2\eta_2+\ldots+\xi_m\eta_m,$$

where† ν is in F, but is not the square of any element in F. The investigation of Γ_m is made in §§ 6, 7 of this paper. The study of Γ_m may be made to depend upon the senary group Γ_2. The structure of the latter may be derived from that of the quaternary hyper-Abelian group. This relation accounts for the treatment of the group Γ_m and the hyper-Abelian group in the same paper.

The hyper-Abelian groups in a finite field were introduced by me in the *Proc. Lond. Math. Soc.*, Vol. XXXI., pp. 30–68. Their structure was there made to depend upon the structure of the hyper-orthogonal groups. The structure of the latter groups in an infinite field presents serious difficulties. The hyper-Abelian groups in an infinite field are consequently investigated *ab initio* in the present paper (§§ 2–5).

---

* If we employ the form ΣX_i^2, we are led to quadratic equations, the discussion of which presents greater difficulties. The desire is to employ only the rational operations of algebra.

† If ν be a square in F, the form is reducible to $x_0y_0+\xi_1\eta_1+\ldots+\xi_m\eta_m$, a case previously mentioned. We thus exclude the case $F\equiv C$.

Consider an arbitrary field F not having the modulus $p = 2$, and containing an element ν not the square of any element in F. In particular, F shall not be the field C of all complex numbers. The equation $X^2 - \nu = 0$ belongs to F, but is irreducible in F. By the adjunction of a root J of this irreducible quadratic equation, the field F is extended to a larger field Q. The elements of Q are therefore of the form $q \equiv \alpha + \beta J$, where α and β belong to F. We set $\bar{q} \equiv \alpha - \beta J$, and call $\bar{q}$ the *conjugate* of q with respect to the field F. In particular, $\bar{J} = -J$, and J, $\bar{J}$ are the two roots of $X^2 - \nu = 0$.

2. *Definition and Generators of the Hyper-Abelian Group.*

Consider the transformation with coefficients in Q,

$$S: \left\{ \begin{aligned} \xi'_i &= \sum_{j=1}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j) \\ \eta'_i &= \sum_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \end{aligned} \right\} \quad (i = 1, \ldots, m).$$

The conditions that S shall leave the function

$$\phi \equiv \sum_{i=1}^{m} \begin{vmatrix} \xi_i & \eta_i \\ \bar{\xi}_i & \bar{\eta}_i \end{vmatrix}$$

formally and absolutely invariant are the following:—

$$(1) \quad \sum_{i=1}^{m} \begin{vmatrix} \alpha_{ij} & \beta_{ij} \\ \bar{\alpha}_{ik} & \bar{\beta}_{ik} \end{vmatrix} = 0, \quad \sum_{i=1}^{m} \begin{vmatrix} \gamma_{ij} & \delta_{ij} \\ \bar{\gamma}_{ik} & \bar{\delta}_{ik} \end{vmatrix} = 0,$$

$$(2) \quad \sum_{i=1}^{m} \begin{vmatrix} \alpha_{ij} & \beta_{ij} \\ \bar{\gamma}_{ik} & \bar{\delta}_{ik} \end{vmatrix} = \begin{matrix} 1 & (j = k) \\ 0 & (j \neq k) \end{matrix},$$

holding for $j, k = 1, \ldots, m$. Hence the inverse of S is

$$S^{-1}: \left\{ \begin{aligned} \xi'_i &= \sum_{j=1}^{m} (\bar{\delta}_{ji}\xi_j - \bar{\gamma}_{ji}\eta_j) \\ \eta'_i &= \sum_{j=1}^{m} (-\bar{\beta}_{ji}\xi_j + \bar{\alpha}_{ji}\eta_j) \end{aligned} \right\} \quad (i = 1, \ldots, m).$$

Writing relations (1) and (2) for S^{-1}, we get

$$(3) \quad \sum_{i=1}^{m} \begin{vmatrix} \alpha_{ki} & \gamma_{ki} \\ \bar{\alpha}_{ji} & \bar{\gamma}_{ji} \end{vmatrix} = 0, \quad \sum_{i=1}^{m} \begin{vmatrix} \beta_{ki} & \delta_{ki} \\ \bar{\beta}_{ji} & \bar{\delta}_{ji} \end{vmatrix} = 0,$$

$$(4) \quad \sum_{i=1}^{m} \begin{vmatrix} \alpha_{ki} & \gamma_{ki} \\ \bar{\beta}_{ji} & \bar{\delta}_{ji} \end{vmatrix} = \begin{matrix} 1 & (j = k) \\ 0 & (j \neq k) \end{matrix}.$$

If σ denote the determinant of S, we note that the determinant of S^{-1} equals $\bar{\sigma}$, so that $\sigma\bar{\sigma} = 1$.

Among the simplest transformations which leave ϕ absolutely invariant, we make use of the following :—*

$$M_i : \xi_i' = \eta_i, \quad \eta_i' = -\xi_i;$$

$$L_{i,a} : \xi_i' = \xi_i + a\eta_i \quad (a \text{ in field } F);$$

$$L'_{i,a} : \eta_i' = \eta_i + a\xi_i \quad (a \text{ in field } F);$$

$$N_{i,j,\tau} : \xi_i' = \xi_i + \tau\eta_j, \quad \xi_j' = \xi_j + \bar{\tau}\eta_i;$$

$$Q_{i,j,\tau} : \xi_i' = \xi_i + \tau\xi_j, \quad \eta_j' = \eta_j - \bar{\tau}\eta_i;$$

$$R_{i,j,\tau} : \eta_i' = \eta_i - \tau\xi_j, \quad \eta_j' = \eta_j - \bar{\tau}\xi_i;$$

$$P_{ij} \quad : \xi_i' = \xi_j, \quad \xi_j' = \xi_i, \quad \eta_i' = \eta_j, \quad \eta_j' = \eta_i;$$

$$T_{i,\tau} \quad : \xi_i' = \tau\xi_i, \quad \eta_i' = \bar{\tau}^{-1}\eta_i.$$

All of these transformations except $T_{i,\tau}$ have determinant unity.

The totality of transformations in the field Q which leave ϕ absolutely invariant constitutes a group called the *hyper-Abelian group* and denoted by $H(2m, Q)$. We proceed to prove that it is generated by the above simple transformations.

Let S be any hyper-Abelian transformation expressed in the above notation. The product $S' \equiv V^{-1}S$ has $a'_{11} \neq 0$, if we take $V = P_{1j}$ or $P_{1j}M_1$ according as $a_{1j} \neq 0$ or $\gamma_{1j} \neq 0$. Then $S' = T_{1,a'_{11}} S_1$, where S_1 is of the above form S with $a_{11} = 1$. The product

$$W \equiv Q_{1,2,a_{12}} N_{1,2,\gamma_{12}} \dots Q_{1,m,a_{1m}} N_{1,m,\gamma_{1m}}$$

leaves η_1 unaltered and replaces ξ_1 by

$$\xi_1 - \eta_1 \sum_{j=2}^{m} \bar{a}_{1j}\gamma_{1j} + \sum_{j=2}^{m} (a_{1j}\xi_j + \gamma\ \eta_j).$$

Hence we may set $S_1 = WS_1'$, where S_1' replaces ξ_1 by

$$\xi_1 + \tau\eta_1 \qquad \left(\tau \equiv \gamma_{11} + \sum_{j=2}^{m} \bar{a}_{1j}\gamma_{1j}\right).$$

* The notations are slight modifications of the standard notations for Abelian linear transformations in the field F. Only the indices actually altered are indicated in the notation.

The first condition (3), for $j = k = 1$, gives

$$\begin{vmatrix} 1 & \tau \\ 1 & \bar{\tau} \end{vmatrix} = 0,$$

so that τ belongs to the field F. We may therefore set

$$S_1' = L_{1,\tau} S'',$$

where S'' belongs to H and does not alter ξ_1. Let S'' replace η_1 by $\sum_{j=1}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j)$. By (4), for $j = k = 1$, we get $\delta_{11} = 1$. Then

$$K \equiv R_{1,2,-\beta_{12}} Q_{2,1,-\delta_{12}} \dots R_{1,m,-\beta_{1m}} Q_{m,1,-\delta_{1m}}$$

leaves ξ_1 fixed and replaces η_1 by

$$\eta_1 + \xi_1 \sum_{j=2}^{m} \bar{\beta}_{1j}\delta_{1j} + \sum_{j=2}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j).$$

We may thus set $S'' = KS_2$, where S_2 belongs to H, leaves ξ_1 unaltered, and replaces η_1 by

$$\eta_1 + \kappa\xi_1 \qquad \left(\kappa \equiv \beta_{11} - \sum_{j=2}^{m} \bar{\beta}_{1j}\delta_{1j}\right).$$

By the second relation (3), for $j = k = 1$, we get $\kappa - \bar{\kappa} = 0$. Hence

$$S_2 = L'_{1,\kappa} S''',$$

where S''' leaves ξ_1 and η_1 fixed. By (3) and (4), for $j = 1$,

$$\gamma_{k1} = \beta_{k1} = 0, \quad \alpha_{k1} = 0 \quad (k = 2, \dots, m).$$

By (4), for $k = 1$, we get $\quad \delta_{j1} = 0 \quad (j = 2, \dots, m)$.

Hence S''' involves only ξ_i, η_i $(i = 2, \dots, m)$.

After m operations similar to that by which S''' was derived from S, we reach a transformation which leaves fixed every variable, and hence is the identity.

Between the generators the following relations hold:—

$$(5) \quad \begin{cases} L'_{i,a} = M_i^{-1} L_{i,-a} M_i, & Q_{i,j,\tau} = M_j N_{i,j,-\tau} M_j^{-1}, \\ R_{i,j,\tau} = M_i^{-1} Q_{i,j,\tau} M_i, & P_{ij} = Q_{j,i,1}^{-1} Q_{i,j,1} Q_{j,i,1}. \end{cases}$$

Hence $H(2m, Q)$ is generated by $M_i, L_{i,a}, N_{i,j,\tau}, T_{i,\tau}$.

In view of the relations

$$(6) \quad \begin{cases} T_{i,\tau} M_i = M\, T_{i,\bar{\tau}^{-1}}, & T_{i,\tau} N_{i,j,\lambda} = N_{i,j,\lambda\tau^{-1}} T_{i,\tau}, \\ T_{i,\tau} L_{i,a} = L_{i,b} T_{i,\tau} & (b = a/\tau\bar{\tau}), \end{cases}$$

any transformation of H can be given the form $hT_{m,\tau}$, when h is derived from the transformations

$$(7) \quad M_i,\ L_{i,a},\ N_{i,j,\tau},\ T_{i,\tau}T_{j,\tau^{-1}},$$

each of determinant unity. But $T_{m,\tau}$ is of determinant unity if, and only if, $\tau = \bar{\tau}$, whence τ belongs to F. But, for any quantity t in the field F, we have

$$(8) \quad T_{i,t} = L'_{i,t} L_{i,-t^{-1}} L'_{i,t} M_i.$$

Hence the transformations of $H(2m, Q)$ which have determinant unity form a sub-group $H_1(2m, Q)$ generated by the transformations (7), *and H_1 is extended to H by the right-hand multipliers** $T_{m,\tau}$.

Hence H_1 is an invariant sub-group of H. This fact also follows from (6), in view of the relation

$$(9) \quad T_{i,\tau}^{-1} M_i T_{i,\tau} = M_i T_{i,\tau\bar{\tau}}.$$

3. *Structure of the Hyper-Abelian Group $H_1(2m, Q)$.*

Every transformation of H_1 is commutative with

$$T_\kappa: \quad \xi'_i = \kappa\xi_i, \quad \eta'_i = \kappa\eta_i \quad (i = 1, 2, \ldots, m),$$

which belongs to H_1 if, and only if,

$$(10) \quad \kappa\bar{\kappa} = 1, \quad \kappa^{2m} = 1.$$

The various transformations T_κ, for which κ satisfies the conditions (10), form an invariant sub-group K of H_1. In order to prove that K is the maximal invariant sub-group of H_1, we show that an invariant sub-group J of H_1, where J contains K without being identical with K, must coincide with H_1.

Let S be a transformation of J not in K. If S be commutative with both $L_{i,1}$ and $L'_{i,1}$ for every $i = 1, 2, \ldots, m$, it has the form

$$S: \quad \xi'_i = a_i\xi_i, \quad \eta'_i = a_i\eta_i \quad (i = 1, \ldots, m),$$

where, by (4), $a_i\bar{a}_i = 1$. Since S' is, by hypothesis, not in K, the a_i are not all equal. If $a_i \neq a_j$, J contains

$$S'' = N_{i,j,1}^{-1} S' N_{i,j,1},$$

which replaces ξ_i by $\qquad a_i\xi + (a_j - a_i)\eta_j$.

* We may restrict τ to elements of Q not in F, such that no two elements τ have a ratio belonging to F, while every element of Q has with some τ a ratio belonging to F.

Since S'' is not of the form S', we may take it in place of the initial S. Suppose therefore that S is not commutative with $L_{1,1}$, for definiteness. Then J contains

$$S_1 = S^{-1} L_{1,1}^{-1} S L_{1,1} \neq I.$$

If S^{-1} replaces η_1 by ω, the product S_1 has the form

$$S_1 : \begin{cases} \xi_1' = \sum_{j=1}^{m} (a_j \xi_j + \gamma_j \eta_j), & \xi_i' = \xi_i - a_{i1}\omega \quad (i = 2, \ldots, m), \\ & \eta_i' = \eta_i - \beta_{i1}\omega \quad (i = 1, \ldots, m), \end{cases}$$

in which the coefficients of ξ_1' need not be determined.

From S_1 we proceed to derive a transformation $\neq I$ and belonging to J and leaving $2m-3$ variables unaltered.* If

$$a_{i1} = \beta_{1i} = 0 \quad (i = 2, \ldots, m),$$

S_1 itself is such a transformation. In the contrary case, the transformed of S_1 by some P_{2j} or $P_{2j} M_2$ will have $a_{21} \neq 0$. Consider therefore S_1 for $a_{21} \neq 0$ and transform it by the product A,

$$R_{1,2,\beta_{11}/a_{21}}\, Q_{3,2,-a_{31}/a_{21}}\, R_{3,2,\beta_{31}/a_{21}} \ldots Q_{m,2,-a_{m1}/a_{21}}\, R_{m,2,\beta_{m1}/a_{21}},$$

which denotes the transformation (where η_2' is not given)

$$A : \begin{cases} \xi_1' = \xi_1, \;\; \xi_2' = \xi_2, \;\; \xi_i' = \xi_i - \dfrac{a_{i1}}{a_{21}}\xi_2 \quad (i = 3, \ldots, m), \\ \eta_i' = \eta_i - \dfrac{\beta_{i1}}{a_{21}}\xi_2 \quad (i = 1, 3, 4, \ldots, m). \end{cases}$$

Then $S_1' \equiv A^{-1} S_1 A$ leaves unaltered η_1, ξ_i, η_i $(i = 3, \ldots, m)$. Transforming S_1' by P_{12}, we obtain in J the transformation (not the identity)

$$S_2 : \begin{cases} \xi_1' = a_{11}\xi_1 + \gamma_{11}\eta_1 + \gamma_{12}\eta_2, & \eta_1' = \beta_{11}\xi_1 + \delta_{11}\eta_1 + \delta_{12}\eta_2, \\ \xi_2' = a_{21}\xi_1 + \gamma_{21}\eta_1 + \xi_2 + \gamma_{22}\eta_2, & \eta_2' = \eta_2, \end{cases}$$

since $a_{12} = \beta_{12} = 0$, $a_{22} = 1$ by the hyper-Abelian relations.†

The group J therefore contains $S_2^{-1} N_{1,2,1}^{-1} S_2 N_{1,2,1}$:

$$S_3 : \begin{cases} \xi_1' = \xi_1 + (1 - a_{11})\,\eta_2, & \eta_1' = \eta_1 - \beta_{11}\eta_2, \\ \xi_2' = \xi_2 + \phi(\xi_1, \eta_1, \eta_2), & \eta_2' = \eta_2. \end{cases}$$

---

* The argument assumes that $m > 1$. For $m = 1$, see the first note in § 4.

† Henceforth we do not write ξ_i, η_i $(i = 3, \ldots, m)$, when not altered.

(*a*) If $\beta_{11}=0$, $a_{11}\neq 1$, then S_3 has the form

$$(11)\quad \begin{cases} \xi_1' = \xi_1+\gamma_{12}\eta_2, & \eta_1' = \eta_1 \\ \xi_2' = a_{21}\xi_1+\gamma_{21}\eta_1+\xi_2+\gamma_{22}\eta_2, & \eta_2' = \eta_2 \end{cases} \quad (\gamma_{12}\neq 0).$$

By the hyper-Abelian conditions $a_{21}=0$, $\gamma_{21}=\bar{\gamma}_{12}$, $\gamma_{22}=\bar{\gamma}_{22}$. Hence S_3 has the form (with $a\neq 0$)

$$(12)\quad \begin{cases} \xi_1' = \xi_1+a\eta_2, & \eta_1' = \eta_1 \\ \xi_2' = \xi_2+\bar{a}\eta_1+\beta\eta_2, & \eta_2' = \eta_2 \end{cases} \quad (\bar{\beta}=\beta).$$

(*b*) If $\beta_{11}\neq 0$, $a_{11}=1$, J contains $M_1^{-1}S_3M_1$, which is of the form (11).

(*c*) If $\beta_{11}\neq 0$, $a_{11}\neq 1$, the transformed of S_3 by $T_{1,\omega}T_{2,\omega^{-1}}$ is

$$S':\begin{cases} \xi_1' = \xi_1+\frac{\omega}{\bar{\omega}}(1-a_{11})\eta_2, & \eta_1' = \eta_1-\frac{\beta_{11}}{\bar{\omega}^2}\eta_2, \\ \xi_2' = \xi_2+\phi_1(\xi_1,\eta_1,\eta_2), & \eta_2' = \eta_2. \end{cases}$$

Then J contains $S'S_3^{-1}$, which replaces ξ_1 and η_1 by, respectively,

$$\xi_1+\left(\frac{\omega}{\bar{\omega}}-1\right)(1-a_{11})\eta_2,\quad \eta_1-(\bar{\omega}^{-2}-1)\beta_{11}\eta_2.$$

Choosing* ω in F different from $0, \pm 1$, we have $\omega=\bar{\omega}$, $\bar{\omega}^2\neq 1$. Transforming the resulting transformation by M_1, we reach a transformation (11).

(*d*) If $\beta_{11}=0$, $a_{11}=1$, S_3 is the identity. Hence S_2 is commutative with $N_{1,2,1}$; so that $\delta_{11}=1$, $a_{21}=\delta_{12}$. By the hyper-Abelian conditions,

$$\bar{\delta}_{12}=-\delta_{12},\quad \gamma_{21}-\bar{\gamma}_{11}\delta_{12}-\bar{\gamma}_{12}=0,\quad \gamma_{11}=\bar{\gamma}_{11}.$$

If $\gamma_{11}=0$, S_2 reduces to the form S_3 and is not the identity. If $\gamma_{11}\neq 0$, we transform S_2 by $Q_{2,1,\tau}$, where $\bar{\tau}\gamma_{11}+\gamma_{12}=0$, and obtain the transformation

$$V:\begin{cases} \xi_1' = \xi_1+\gamma_{11}\eta_1, & \eta_1' = \eta_1+\delta_{12}\eta_2, \\ \xi_2' = \xi_2+\delta_{12}\xi_1+\gamma_{21}'\eta_1+\gamma_{22}'\eta_2, & \eta_2' = \eta_2, \end{cases}$$

where $\gamma_{11}=\bar{\gamma}_{11}$, $\bar{\delta}_{12}=-\delta_{12}$, $\gamma_{21}'=\gamma_{11}\delta_{12}$. If $\delta_{12}\neq 0$, we consider

$$V^{-1}T_{2,-1}^{-1}VT_{2,-1}:\begin{cases} \xi_1' = \xi_1, & \eta_1' = \eta_1-2\delta_{12}\eta_2, \\ \xi_2' = \xi_2+\Theta(\xi_1,\eta_1,\eta_2), & \eta_2' = \eta_2, \end{cases}$$

---

* If F is of finite order p^n, the choice is valid if $p^n>3$.

which is of the form S_3 above, and is not the identity.* If $\delta_{12}=0$, V is of the form $L_{1,\gamma_1}L_{2,\gamma_2}$ with $\gamma_1 \neq 0$. Transforming it by $T_{1,\mu}$, μ belonging to the field F, we obtain $L_{1,\mu^2\gamma_1}L_{2,\gamma_2}$. Multiplying it by the inverse of a similar transformation, we get in J the transformation $L_{1,\tau}$, $\tau \equiv \gamma_1(\mu^2-\mu_1^2)$. Taking

$$\mu = \tfrac{1}{2}(\rho+1), \quad \mu_1 = \tfrac{1}{2}(\rho-1),$$

we get $\tau = \gamma_1\rho$. By proper choice of ρ, we can make τ assume any given value in the field F.

It remains to discuss the transformation (12), in which α and β are not both zero. If $\alpha = 0$, it is $L_{2,\beta}$. If $\alpha \neq 0$, it is transformed by $Q_{2,1,\tau}$ into the transformation

$$\begin{cases} \xi_1' = \xi_1+\alpha\eta_2, & \eta_1' = \eta_1, \\ \xi_2' = \xi_1+\bar{\alpha}\eta_1+(\beta+\tau\alpha+\bar{\tau}\bar{\alpha})\,\eta_2, & \eta_2' = \eta_2. \end{cases}$$

Taking τ so that $\tau\alpha$ is a root of $x+\bar{x} = -\beta$, an equation solvable† in Q since β belongs to F, we reach in the group J the transformation $N_{1,2,\alpha}$. But

$$T_{1,\tau}^{-1}N_{1,2,\alpha}T_{1,\tau} = N_{1,2,\alpha\tau}, \quad T_{2,\tau}^{-1}N_{1,2,\alpha}T_{2,\tau} = N_{1,2,\alpha\bar{\tau}}.$$

Since $T_{1,\tau\bar{\tau}}T_{2,\tau^{-1}}$ has determinant unity, it belongs to H_1. It transforms $N_{1,2,\alpha}$ into $N_{1,2,\alpha\tau^2}$, which therefore belongs to J. Hence J contains $N_{1,2,\rho}$, $\rho \equiv \alpha(\tau_1^2-\tau_2^2)$, where ρ can be made to assume any given value in the field Q. Having $N_{1,2,1}$, J contains

$$(13) \quad L_{1,1} = (M_2L_{2,1})^{-1}N_{1,2,1}(M_2L_{2,1})M_2^{-1}N_{1,2,1}^{-1}M.N_{1,2,1}.$$

Hence, in every case, J contains the transformation $L_{1,\tau}$, where τ is arbitrary in the field F. Then J contains

$$M_1 \equiv L_{1,1}\,.\,M_1^{-1}L_{1,1}M_1\,.\,L_{1,1}.$$

Transforming by P_{1i}, we reach every M_i and $L_{i,\tau}$. Then, by (13), J contains $N_{1,2,1}$. As above, we reach in J every $N_{1,2,\rho}$, ρ in Q. Then, by (5), J contains every $N_{i,j,\rho}$, $Q_{i,j,\rho}$, $R_{i,j,\rho}$, P_{ij}. Since

$$R_{i,j,\mu^{-1}}N_{i,j,\bar{\mu}}R_{i,j,\mu^{-1}} = M_iM_jP\ \ T_{i,\bar{\mu}}T_{j,\mu},$$

J contains every $T_{i,\bar{\mu}}T_{j,\mu}$, μ in the field Q. But, by (8), J contains

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* If there be a modulus p, we assume that $p \neq 2$.

† If the modulus p, when existent, differs from 2.

every $T_{i,t}$, t in F. Hence, since $\mu\bar{\mu}$ belongs to F, J contains every $T_{i,\mu^{-1}} T_{j,\mu}$. Since J therefore contains every generator (7) of H_1, we conclude that $J \equiv H_1$. Except for the case when Q has a modulus 2, we have proved the theorem :

The maximal invariant sub-group of the group $H_1(2m, Q)$ of hyper-Abelian transformations of determinant unity is composed of the transformations T_κ, which multiply every variable by the same factor κ. The group of the transformations of H_1, when taken fractionally, is simple.

4. *The Abelian Linear Group.*

Those transformations of $H(2m, Q)$ whose coefficients all belong to the included field F form a sub-group, *the general Abelian linear group in the field F*. It is designated $GA(2m, F)$. The Abelian transformations of determinant unity form a sub-group* called the *special Abelian linear group in the field F*, $SA(2m, F)$. By the proper specialization of the above developments, we obtain the theorem :—

The special Abelian group $SA(2m, F)$ is generated by the transformations M_i, $L_{i,a}$, $N_{i,j,a}$, where a is arbitrary in F. Its maximal invariant sub-group is composed of the identity and the transformation T_{-1}, which changes the sign of every variable.

The maximal sub-group of $H(2m, Q)$ within which $SA(2m, F)$ is self-conjugate may be determined as in the author's paper in the *Proc. Lond. Math. Soc.* (Vol. xxxi., pp. 34–40). We derive the theorem :—

Every transformation of $H(2m, Q)$ which transforms the group $SA(2m, F)$ into itself may be expressed as a product UV_α, where U belongs to $SA(2m, F)$ and V_α denotes the transformation

$$T_{1,\alpha} T_{2,\alpha} \ldots T_{m,\alpha} : \; \xi_i' = \alpha\xi_i, \quad \eta_i' = \bar{\alpha}^{-1}\eta_i \quad (i = 1, \ldots, m).$$

We may restrict α to those elements r of Q of which no two have as ratio an element in F, while every element of Q has with some r a ratio belonging to F.

---

* For $m = 1$, they are identical: $H_1(2, Q) \equiv SA(2, F)$. Each is the group of all binary transformations in F of determinant unity. It is shown in the *Trans. Amer. Math. Soc.* (*l.c.*, § 1) that the structure in the case $m = 1$ follows the same law as in the case $m > 1$.

5. *Structure of a certain Senary Linear Group.*

The preceding results concerning the structure of the quaternary hyper-Abelian group $H_1(4, Q)$ in the field Q may be applied to determine the structure of a senary group in the field F. The latter group plays a fundamental *rôle* in the later investigation of a general class of linear groups defined by a quadratic invariant.

We consider the second compound* of the group $H_1(4, Q)$ and obtain a senary group in Q with the absolute invariant†

$$F_4 \equiv Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23}.$$

In the second compound, we introduce new variables‡

$$\xi_0 = \tfrac{1}{2}(Y_{12} - Y_{34}), \quad \eta_0 = \frac{-1}{2J}(Y_{12} + Y_{34}),$$

$$\xi_1 = Y_{13}, \quad \eta_1 = Y_{24}, \quad \xi_2 = -Y_{14}, \quad \eta_2 = Y_{23}.$$

The invariant F_4 is changed into the negative of the function

$$(14) \quad \xi_0^2 - \nu\eta_0^2 + \xi_1\eta_1 + \xi_2\eta_2,$$

where $\nu = J^2$ (see § 1). Hence to the group H_1 corresponds a senary group G_2 with the invariant (14). To the transformation $T_{1,\omega}T_{2,\omega}^{-1}$ of H_1 corresponds

$$(15) \quad \begin{cases} \xi_0' = \lambda\xi_0 + \mu\eta_0, & \xi_1' = \xi_1, \quad \xi_2' = \omega\bar{\omega}\xi_2, \\ \eta_0' = \dfrac{1}{\nu}\mu\xi_0 + \lambda\eta_0, & \eta_1' = \eta_1, \quad \eta_2' = (\omega\bar{\omega})^{-1}\eta_2, \end{cases}$$

where

$$(16) \quad \lambda = \tfrac{1}{2}\left(\frac{\bar{\omega}}{\omega} + \frac{\omega}{\bar{\omega}}\right), \quad \mu = \frac{J}{2}\left(\frac{\bar{\omega}}{\omega} - \frac{\omega}{\bar{\omega}}\right).$$

Since $\bar{J} = -J$, it follows that λ and μ belong to the field F.

Now G_2 contains $\xi_2' = r\xi_2$, $\eta_2' = r^{-1}\eta_2$ if, and only if, r is a square in the field F. Hence G_2 contains

$$(17) \quad [\lambda, \mu]: \begin{Bmatrix} \xi_0' = \lambda\xi_0 + \mu\eta_0 \\ \eta_0' = \dfrac{1}{\nu}\mu\xi_0 + \lambda\eta_2 \end{Bmatrix} \quad \left(\lambda^2 - \frac{1}{\nu}\mu^2 = 1\right)$$

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* *Proc. Lond. Math. Soc.*, Vol. xxx., p. 81.

† For the case of finite fields, see *Bulletin Amer. Math. Soc.*, Vol. vi., p. 323.

‡ If there be a modulus p, we assume henceforth that $p \neq 2$.

if, and only if, $\omega\bar{\omega}$ is a square in F. By (16), we get

$$\lambda+\mu/J=\bar{\omega}/\omega, \quad \lambda-\mu/J=\omega/\bar{\omega}.$$

Then, if $\omega\bar{\omega}=t^2$, where t is in F, we may set

$$\frac{\omega}{t}=\alpha-\frac{\beta}{J}, \quad \frac{\bar{\omega}}{t}=\alpha+\frac{\beta}{J} \quad (\alpha,\ \beta \text{ in } F).$$

Hence, by taking the product of the two,

$$\alpha^2-\beta^2/J^2 \equiv \alpha^2-\frac{1}{\nu}\beta^2=1.$$

Since $\omega/\bar{\omega}=\omega^2/t^2$, we get

$$\lambda=\alpha^2+\frac{1}{\nu}\beta^2\equiv 2\alpha^2-1, \quad \mu=2\alpha\beta.$$

Hence, if G_2 contains $[\lambda,\ \mu]$, the latter is of the form

$$(18) \quad [\alpha,\ \beta]^2: \left\{\begin{aligned} \xi_0' &= (2\alpha^2-1)\,\xi_0+2\alpha\beta\eta_0 \\ \eta_0' &= \frac{2}{\nu}\alpha\beta\xi_0+(2\alpha^2-1)\,\eta_0 \end{aligned}\right\} \quad \left(\alpha^2-\frac{1}{\nu}\beta^2=1\right).$$

We next show that, inversely, every $[\alpha,\ \beta]^2$ belongs to G_2. We set $\lambda=2\alpha^2-1$, $\mu=2\alpha\beta$, and show that equations (16) can be satisfied by an element ω which is a square in Q, so that $\omega\bar{\omega}$ is a square in F. The conditions give

$$\frac{\bar{\omega}}{\omega}=\lambda+\frac{\mu}{J}=\left(\alpha+\frac{\beta}{J}\right)^2, \quad \frac{\omega}{\bar{\omega}}=\lambda-\frac{\mu}{J}=\left(\alpha-\frac{\beta}{J}\right)^2.$$

The latter follows from the former, since

$$\left(\alpha+\frac{\beta}{J}\right)\left(\overline{\alpha+\frac{\beta}{J}}\right)=\left(\alpha+\frac{\beta}{J}\right)\left(\alpha-\frac{\beta}{J}\right)=\alpha^2-\frac{\beta^2}{J^2}=\alpha^2-\frac{1}{\nu}\beta^2=1.$$

Let $\alpha+\beta/J\equiv w$, so that $\bar{w}=1/w$. We seek the solutions ω of

$$\frac{\bar{\omega}}{\omega}=w^2=\frac{w}{\bar{w}}.$$

We may take $\omega=r\bar{w}$, where r is any element $\neq 0$ in F. Since

$$\frac{2}{\alpha+1}\left(\alpha+\frac{\beta}{J}\right)\equiv\left(1+\frac{\beta}{(\alpha+1)\,J}\right)^2,$$

the value $r=2/(\alpha+1)$, for $\alpha\neq -1$, makes rw, and hence also $r\bar{w}$, a square in Q. For $\alpha=-1$, then $\beta=0$ and $[\alpha,\ \beta]^2$ is the identity.

Hence G_2 contains $[\lambda, \mu]$ if, and only if, it be of the form (18).

To the transformations M_1, M_2, $L_{1,\lambda}$, $L_{2,\lambda}$, $N_{1,2,\lambda}$ of H_1, λ being in F, there correspond the respective transformations of G_2:

$$S_{12}P_{12}T_{2,-1}, \quad P_{12}T_{1,-1}, \quad W_{1,2,\lambda}, \quad Q_{1,2,-\lambda}, \quad X_{0,1,\lambda},$$

and to $N_{1,2,\tau}$, where $\bar{\tau} = -\tau$, corresponds $U_{0,1,\lambda}$, where $\lambda \equiv J\tau$ belongs to F. The notations employed have the following definitions:—

$$\begin{array}{lll}
W_{i,j,\lambda}: & \xi_i' = \xi_i + \lambda\eta_j, & \xi_j' = \xi_j - \lambda\eta_i; \\
V_{i,j,\lambda}: & \eta_i' = \eta_i + \lambda\xi_j, & \eta_j' = \eta_j - \lambda\xi_i; \\
Q_{i,j,\lambda}: & \xi_i' = \xi_i + \lambda\xi_j, & \eta_j' = \eta_j - \lambda\eta_i; \\
X_{0,j,\lambda}: & \xi_0' = \xi_0 - \lambda\eta_j, & \xi_j' = \xi_j + 2\lambda\xi_0 - \lambda^2\eta_j; \\
Y_{0,j,\lambda}: & \xi_0' = \xi_0 - \lambda\xi_j, & \eta_j' = \eta_j + 2\lambda\xi_0 - \lambda^2\xi_j; \\
U_{0,j,\lambda}: & \eta_0' = \eta_0 + \dfrac{\lambda}{\nu}\eta_j, & \xi_j' = \xi_j + 2\lambda\eta_0 + \dfrac{\lambda^2}{\nu}\eta_j; \\
Z_{0,j,\lambda}: & \eta_0' = \eta_0 + \dfrac{\lambda}{\nu}\xi_j, & \eta_j' = \eta_j + 2\lambda\eta_0 + \dfrac{\lambda^2}{\nu}\xi_j; \\
T_{i,\lambda}: & \xi_i' = \lambda\xi_i, & \eta_i' = \lambda^{-1}\eta_i;
\end{array}$$

$$P_{ij} = (\xi_i\xi_j)(\eta_i\eta_j); \quad S_{ij} = (\xi_i\eta_i)(\xi_j\eta_j).$$

Between these transformations exist the relations

$$(19) \quad V_{i,j,\lambda} = S_{ij}^{-1}W_{i,j,\lambda}S_{ij}, \quad Q_{i,j,\lambda} = S_{jk}^{-1}W_{i,j,\lambda}S_{jk},$$

$$(20) \quad Y_{0,j,\lambda} = S_{jk}^{-1}X_{0,j,\lambda}S_{jk}, \quad Z_{0,j,\lambda} = S_{j,k}^{-1}U_{0,j,\lambda}S_{jk}.$$

Since the group $H_1(4, Q)$ is generated by the transformations (7), we may state the earlier result in the following form:—

There exists a senary linear group G_2 in the field F, which leaves the function (14) *absolutely invariant, and is generated by S_{12}, $P_{12}T_{2,-1}$, $W_{1,2,\lambda}$, $Q_{1,2,\lambda}$, $X_{0,1,\lambda}$, and* (15). *It contains the transformations* (18) *and $U_{0,1,\lambda}$. If -1 is a square in F or if -1 and $-\nu$ are both not-squares in F, then G_2 is a simple group. If -1 is a not-square and $-\nu$ is a square in F, then G_2 has the maximal invariant sub-group $\{I, T\}$, where T changes the signs of the six variables.*

To verify the last statement, we must determine the transformations T_κ, where $\kappa^4 = 1$, $\kappa\bar{\kappa} = 1$. Let i be a root of $x^2 = -1$. There are three possible cases.

(1) If -1 is a square in F, then i belongs to F and $\bar{i} = i$; so that $i\bar{i} = i^2 = -1$. Hence κ cannot be $\pm i$.

(2) If -1 is a not-square and $-\nu$ is a square in F, then $-\nu$ is a square in Q and ν is the square of J in Q; so that -1 is a square in Q. Hence i is in Q, but not in F. Let

$$i = x + Jy \quad (x, y \text{ in } F, y \neq 0);$$

therefore
$$-1 = x^2 + \nu y^2 + 2xyJ;$$

therefore
$$xy = 0, \quad x^2 + \nu y^2 = -1.$$

Hence $x = 0$ and $-\nu = (\nu y)^2$. Hence

$$i = Jy, \quad \bar{i} = -Jy, \quad i\bar{i} = 1.$$

It follows that κ may be ± 1 or $\pm i$.

(3) If -1 and $-\nu$ are not-squares in F, then i is not in Q. For, if $i = x + Jy$, x and y being in F, then $y \neq 0$, since i is not in F. As in case (2), $x = 0$, $-\nu = (\nu y)^2 =$ square in F. This being contrary to hypothesis, κ cannot be $\pm i$.

In cases (1) and (3), $\kappa = \pm 1$; so that every T_κ corresponds in the second compound to the identity.

6. *Definition and Generators of the Group* Γ_m.

We study the group Γ_m of all transformations in F,

$$(21) \qquad S: \left\{ \begin{aligned} \xi_i' &= \sum_{j=0}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j) \\ \eta_i' &= \sum_{j=0}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \end{aligned} \right\} \quad (i = 0, 1, \ldots, m),$$

which leave formally and absolutely invariant

$$\phi_m \equiv \xi_0^2 - \nu\eta_0^2 + \xi_1\eta_1 + \xi_2\eta_2 + \ldots + \xi_m\eta_m,$$

ν being a not-square in F. The conditions on S are

$$(22) \qquad \alpha_{0j}^2 - \nu\beta_{0j}^2 + \sum_{i=1}^{m} \alpha_{ij}\beta_{ij} = \begin{cases} 1 & (j = 0) \\ 0 & (j = 1, \ldots, m), \end{cases}$$

$$(23) \qquad \gamma_{0j}^2 - \nu\delta_{0j}^2 + \sum_{i=1}^{m} \gamma_{ij}\delta_{ij} = \begin{cases} -\nu & (j = 0) \\ 0 & (j = 1, \ldots, m), \end{cases}$$

$$\left.\begin{aligned} &(24) \quad 2\alpha_{0j}\alpha_{0k} - 2\nu\beta_{0j}\beta_{0k} + \sum_{i=1}^{m} (\alpha_{ij}\beta_{ik} + \alpha_{ik}\beta_{ij}) = 0 \\ &(25) \quad 2\gamma_{0j}\gamma_{0k} - 2\nu\delta_{0j}\delta_{0k} + \sum_{i=1}^{m} (\gamma_{ij}\delta_{ik} + \gamma_{ik}\delta_{ij}) = 0 \end{aligned}\right\} \begin{pmatrix} j, k = 0, 1, \ldots, m \\ j \neq k \end{pmatrix},$$

$$(26) \quad 2\alpha_{0j}\gamma_{0k} - 2\nu\beta_{0j}\delta_{0k} + \sum_{i=1}^{m} (\alpha_{ij}\delta_{ik} + \gamma_{ik}\beta_{ij}) = \begin{cases} 1 & (j = k \neq 0) \\ 0 & (j = k = 0, \text{ or } j \neq k) \end{cases}$$
$$(j, k = 0, 1, \ldots, m).$$

In view of these relations, the inverse of S is

$$S^{-1}: \quad \begin{cases} \xi_0' = \alpha_{00}\xi_0 - \nu\beta_{00}\eta_0 + \tfrac{1}{2}\sum\limits_{j=1}^{m}(\beta_{j0}\xi_j + \alpha_{j0}\eta_j), \\ \eta_0' = -\dfrac{1}{\nu}\gamma_{00}\xi_0 + \delta_{00}\eta_0 - \dfrac{1}{2\nu}\sum\limits_{j=1}^{m}(\delta_{j0}\xi_j + \gamma_{j0}\eta_j), \\ \xi_i' = 2\gamma_{0i}\xi_0 - 2\nu\delta_{0i}\eta_0 + \sum\limits_{j=1}^{m}(\delta_{ji}\xi_j + \gamma_{ji}\eta_j), \\ \eta_i' = 2\alpha_{0i}\xi_0 - 2\nu\beta_{0i}\eta_0 + \sum\limits_{j=1}^{m}(\beta_{ji}\xi_j + \alpha_{ji}\eta_j) \end{cases}$$

$$(i = 1, \ldots, m).$$

Writing relations (22)-(26) for the transformation S^{-1}, we get

$$(27) \quad \alpha_{j0}^2 - \frac{1}{\nu}\gamma_{j0}^2 + 4\sum_{i=1}^{m}\alpha_{ji}\gamma_{ji} = \begin{cases} 1 & (j=0), \\ 0 & (j=1, \ldots, m), \end{cases}$$

$$(28) \quad \beta_{j0}^2 - \frac{1}{\nu}\delta_{j0}^2 + 4\sum_{i=1}^{m}\beta_{ji}\delta_{ji} = \begin{cases} -1/\nu & (j=0), \\ 0 & (j=1, \ldots, m), \end{cases}$$

$$\left.\begin{aligned} &(29) \quad \tfrac{1}{2}\beta_{j0}\beta_{k0} - \frac{1}{2\nu}\delta_{j0}\delta_{k0} + \sum_{i=1}^{m}(\delta_{ji}\beta_{ki} + \delta_{ki}\beta_{ji}) = 0 \\ &(30) \quad \tfrac{1}{2}\alpha_{j0}\alpha_{k0} - \frac{1}{2\nu}\gamma_{j0}\gamma_{k0} + \sum_{i=1}^{m}(\gamma_{ji}\alpha_{ki} + \gamma_{ki}\alpha_{ji}) = 0 \end{aligned}\right\} \quad \begin{pmatrix} j, k = 0, 1, \ldots, m \\ j \neq k \end{pmatrix},$$

$$(31) \quad \tfrac{1}{2}\beta_{j0}\alpha_{k0} - \frac{1}{2\nu}\delta_{j0}\gamma_{k0} + \sum_{i=1}^{m}(\delta_{ji}\alpha_{ki} + \beta_{ji}\gamma_{ki}) = \begin{cases} 1 & (j = k \neq 0), \\ 0 & (j = k = 0, \text{ or } j \neq k), \end{cases}$$

$$(j, k = 0, 1, \ldots, m).$$

The transformations of determinant unity in Γ_m form an invariant subgroup Γ_m' of index 2, generated by*

$$(32) \quad \begin{cases} W_{i,j,\lambda},\ V_{i,j,\lambda},\ Q_{i,j,\lambda},\ X_{0,j,\lambda},\ Y_{0,j,\lambda},\ U_{0,j,\lambda},\ Z_{0,j,\lambda}, \\ T_{i,\lambda},\ S_{ij},\ P_{ij} \quad \textit{and} \quad [\lambda, \mu]. \end{cases}$$

Let S be an arbitrary transformation of Γ_m and let it replace ξ_1 by

$$f_1 \equiv \sum_{j=0}^{m}(\alpha_{1j}\xi_j + \gamma_{1j}\eta_j),$$

---

* If H is a sub-group of an infinite group G, and if the operators of G may be expressed uniquely in the form hS_i, where h belongs to H and where the S_i are not in H, the number of the S_i is called the index of H under G.

where, by (27), for $j = 1$,

$$(33) \quad \alpha_{10}^2 - \frac{1}{\nu}\gamma_{10}^2 + 4\sum_{i=1}^{m} \alpha_{1i}\gamma_{1i} = 0.$$

We proceed to determine a transformation Σ which replaces ξ_1 by f_1 and is derived from the transformations (32). We note that α_{1i}, γ_{1i} $(i = 1, \ldots, m)$ are not all zero, since otherwise $\alpha_{10} = \gamma_{10} = 0$ by (33); so that $f_1 \equiv 0$. Three cases arise :—

(a) If $\alpha_{11} \neq 0$, we may take as Σ the product

$$\Sigma \equiv T_{1,\alpha_{11}} U_{0,1,\frac{1}{2}\gamma_{10}} X_{0,1,\frac{1}{2}\alpha_{10}} K,$$

in which K denotes the transformation

$$K \equiv Q_{1,2,\alpha_{12}} W_{1,2,\gamma_{12}} \cdots Q_{1,m,\alpha_{1m}} W_{1,m,\gamma_{1m}};$$

so that K replaces ξ_1 by

$$\xi_1 - \eta_1 \sum_{j=2}^{m} \alpha_{1j}\gamma_{1j} + \sum_{j=2}^{m} (\alpha_{1j}\xi_j + \gamma_{1j}\eta_j).$$

In view of (33), we see that Σ replaces ξ_1 by f_1.

(b) If $\gamma_{11} \neq 0$, we may choose for Σ the product

$$S_{12} T_{1,\gamma_{11}} U_{0,1,\frac{1}{2}\gamma_{10}} X_{0,1,\frac{1}{2}\alpha_{10}} K',$$

where K' is derived from K by interchanging α_{12} with γ_{12}.

(c) If $\alpha_{1j} = \gamma_{1j} = 0$ $(j = 1, \ldots, s-1)$, while α_{1s} and γ_{1s} are not both zero, we obtain by case (a) or case (b) a transformation Σ' which replaces ξ_s by f_1. Then $\Sigma \equiv \Sigma' P_{1s}$ replaces ξ_1 by f_1.

We may therefore set $S = \Sigma S'$, where S' is a transformation of Γ_m which leaves ξ_1 unaltered. Let S' replace η_1 by

$$f_1' \equiv \sum_{j=0}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j),$$

where, by (31), for $j = k = 1$, $\delta_{11} = 1$, and by (28), for $j = 1$,

$$(34) \quad \tfrac{1}{4}\beta_{10}^2 - \frac{1}{4\nu}\delta_{10}^2 + \sum_{i=1}^{m} \beta_{1i}\delta_{1i} = 0.$$

It follows that the product Σ_1 :

$$Z_{0,1,\frac{1}{2}\delta_{10}} Y_{0,1,\frac{1}{2}\beta_{10}} V_{2,1,-\beta_{12}} Q_{2,1,-\delta_{12}} \cdots V_{m,1,-\beta_{1m}} Q_{m,1,-\delta_{1m}}$$

leaves ξ_1 fixed and replaces η_1 by f_1'. We may set

$$S' = \Sigma_1 S_1, \quad S \equiv \Sigma\Sigma_1 S_1,$$

where S_1 is a transformation of Γ_m which leaves ξ_1 and η_1 unaltered. If S_1 be written in the form (21), it has

$$\gamma_{1j} = \beta_{1j} = 0 \quad (j = 0, 1, \ldots, m),$$

$$\alpha_{11} = \delta_{11} = 1, \quad \alpha_{1j} = \delta_{1j} = 0 \quad (j = 0, 2, \ldots, m).$$

Then, by (29), (30), and (31), for $j = 1$, and by (31), for $k = 1$,

$$\beta_{k1} = \gamma_{k1} = \alpha_{k1} = \delta_{j1} = 0 \quad (j, k = 0, 2, \ldots, m).$$

Hence S_1 is a transformation of Γ_m which involves only

$$\xi_i, \quad \eta_i \quad (i = 0, 2, \ldots, m).$$

We proceed with S_1 as we did with S. After $m-1$ such steps, we reach a transformation S_{m-1} which involves only the variables $\xi_0, \eta_0, \xi_m, \eta_m$. Transforming it by P_{1m}, we reach a transformation S involving only $\xi_0, \eta_0, \xi_1, \eta_1$. Let it replace ξ_1 by

$$f \equiv \alpha_{10}\xi_0 + \gamma_{10}\eta_0 + \alpha_{11}\xi_1 + \gamma_{11}\eta_1.$$

If $\alpha_{11} \neq 0$, we proceed as in case (a), taking $K = I$. If $\alpha_{11} = 0$, then $\alpha_{10} = \gamma_{10} = 0$ by (33); so that $\gamma_{11} \neq 0$. We may therefore set

$$S = (\xi_1\eta_1)\, T_{1, \gamma_{11}} L,$$

where L does not alter ξ_1. Let L replace η_1 by

$$f' \equiv \beta_{10}\xi_0 + \delta_{10}\eta_0 + \beta_{11}\xi_1 + \delta_{11}\eta_1,$$

where, by (31), for $j = k = 1$, and by (28), for $j = 1$,

$$\delta_{11} = 1, \quad \tfrac{1}{4}\beta_{10}^2 - \frac{1}{4\nu}\delta_{10}^2 + \beta_{11}\delta_{11} = 0.$$

Hence $$L = Z_{0, 1, \frac{1}{2}\delta_{10}}\, Y_{0, 1, \frac{1}{2}\beta_{10}}\, L',$$

where L' does not alter ξ_1 and η_1, and therefore involves only ξ_0 and η_0. Let L' replace ξ_0 by $\alpha_{00}\xi_0 + \gamma_{00}\eta_0$, where, by (27),

$$\alpha_{00}^2 - \frac{1}{\nu}\gamma_{00}^2 = 1.$$

Hence $L' = [\alpha_{00}, \gamma_{00}]\, L''$, where L'' leaves ξ_0 fixed and replaces η_0 by $\beta_{00}\xi_0 + \delta_{00}\eta_0$. By (31), for $j = k = 1$, $\beta_{00} = 0$. Then, by (28), for $j = 0$, $\delta_{00}^2 = 1$. Hence L'' is either the identity or else Y_0, which alters only η_0, whose sign it changes. But

$$Y_0 \equiv U_{0, j, -\nu}\, Z_{0, j, 1}\, U_{0, j, -\nu}\, T_{j, \nu^{-1}}\, (\xi_j \eta_j).$$

It follows that any transformation of Γ_m may be given one of the

two forms: A, $A\,(\xi_m\,\eta_m)$, where A is derived from the transformations (32), all of determinant unity.

7. *Definition and Structure of the Group G_m.*

In view of § 5, the senary group Γ_2' contains a sub-group G_2 which contains the transformations

$$S_{12},\ P_{12}T_{2,-1},\ W_{1,2,\lambda},\ Q_{1,2,\lambda},\ X_{0,1,\lambda},\ U_{0,1,\lambda},\ T_{1,\lambda}T_{2,\lambda},\ T_{1,\lambda^2},$$

together with (15) and (18), but does not contain $T_{1,\mu}$ or $T_{2,\mu}$, when μ is a not-square in the field F.

Denote by G_m the sub-group of Γ_m' obtained by extending G_2 by S_{ij} $(i, j = 1, \ldots, m)$. In view of the formulæ

$$T_{i,\lambda}S_{ij} = S_{ij}T_{i,\lambda^{-1}},\quad T_{i,\lambda}P_{ij} = P_{ij}T_{j,\lambda},$$

$$T_{i,\lambda}Y_{0,i,\tau} = Y_{0,i,\tau\lambda}T_{i,\lambda},\quad T_{i,\lambda}Z_{0,i,\tau} = Z_{0,i,\tau\lambda}T_{i,\lambda},$$

it follows that every transformation of Γ_m' may be expressed as a product $BT_{m,\lambda}$, where B belongs to G_m.

G_m is an invariant sub-group of Γ_m' and is extended to the latter by the right-hand multipliers $T_{m,\mu}$, where μ runs through the series of those not-squares in the field F the ratio of no two of which is a square in F.

Let K be an invariant sub-group of G_m which contains a transformation S, neither the identity I nor the transformation T which changes the sign of every variable.* Let $m \geqq 3$.

LEMMA I.—*The group K contains a transformation which multiplies ξ_1 by a constant and is neither I nor T.*

Let the given transformation S replace ξ_1 by

$$f_1 \equiv \sum_{j=0}^{m} (a_{1j}\xi_j + \gamma_{1j}\eta_j)\quad [\text{subject to (33)}].$$

(*a*) If $\gamma_{11} \neq 0$, G_m contains the product P:

$$T_{1,\gamma_{11}^{-1}}\,T_{2,\gamma_{11}^{-1}}\,Z_{0,1,\frac{1}{2}\gamma_{10}}\,Y_{0,1,\frac{1}{2}a_{10}}\,V_{2,1,-\gamma_{11}a_{12}}\,Q_{2,1,-\gamma_{11}^{-1}\gamma_{12}}B,$$

$$B \equiv V_{3,1,-a_{13}}\,Q_{3,1,-\gamma_{13}}\cdots V_{m,1,-a_{1m}}\,Q_{m,1,-\gamma_{1m}}.$$

We find that P replaces ξ_1 by $\gamma_{11}^{-1}\xi_1$, and η_1 by f_1. Hence K contains $S_1 \equiv P^{-1}SP$, which replaces ξ_1 by $\gamma_{11}^{-1}\eta_1$.

If S_1 multiplies ξ_2 by a constant, its transformed by $P_{12}T_{2,-1}$ yields a

* In certain cases T belongs to G_m; in other cases it does not.

[redacted]

transformation in K which multiplies ξ_1 by a constant and is neither I nor T.

If S_1 does not multiply ξ_2 by a constant, there exists* in G_m a transformation C which leaves ξ_1 and η_1 fixed, and is not commutative with S_1; so that K contains $S_1^{-1}C^{-1}S_1C$, which leaves ξ_1 unaltered and is neither I nor T.

(*b*) If $\gamma_{11} = 0$, we may take $a_{12} \neq 0$. In fact, if

$$a_{12} = a_{13} = \ldots = a_{1m} = 0,$$

then $a_{10} = \gamma_{10} = 0$ by (33), while the γ_{1j} are not all zero; but for $\gamma_{12} \neq 0$, for example, the transformed of S by S_{23} has $a'_{12} \neq 0$. Setting

$$B_1 \equiv Q_{2,\,4,\,a_{14}} W_{2,\,4,\,\gamma_{14}} \cdots Q_{2,\,m,\,a_{1m}} W_{2,\,m,\,\gamma_{1m}},$$

we find that G_m contains the product R:

$$T_{2,\,a_{12}} T_{3,\,a_{12}} U_{0,\,2,\,\frac{1}{2}\gamma_{10}} X_{0,\,2,\,\frac{1}{2}a_{10}} Q_{2,\,1,\,a_{11}} Q_{2,\,3,\,a_{12}^{-1}a_{13}} W_{2,\,3,\,a_{12}\gamma_{13}} B_1,$$

which replaces ξ_2 by f_1, without altering ξ_1. Hence K contains $S_1 = R^{-1}SR$, which replaces ξ_1 by ξ_2. It follows that K contains a transformation, neither I nor T, which leaves ξ_1 and η_1 unaltered.†

LEMMA II.—*The group K contains a transformation which does not alter ξ_1 or η_1, and is different from the identity.*

In view of Lemma I., K contains a transformation S, neither I nor T, which replaces ξ_1 by $a\xi_1$, and η_1 by

$$\sum_{j=0}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j).$$

By (31), for $j = k = 1$, $\delta_{11} = a^{-1}$. By (28), for $j = 1$,

$$(34) \qquad \tfrac{1}{4}\beta_{10}^2 - \frac{1}{4\nu}\delta_{10}^2 + \sum_{i=1}^{m} \beta_{1i}\delta_{1i} = 0.$$

(*a*) Let $\beta_{11} = 0$, $\beta_{1j} = \delta_{1j} = 0$ $(j = 2, \ldots, m)$. Then $\beta_{10} = \delta_{10} = 0$ by (34). Hence $S = T_{1,a} S_1$, where S_1 leaves ξ_1 and η_1 unaltered. As in § 6, case (*c*), S_1 involves only the variables

$$\xi_0, \quad \xi_j, \quad \eta_j \quad (j = 2, \ldots, m).$$

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* We may take $C = V_{2,3,\lambda}$ or $Q_{3,2,\lambda}$. See *Proc. Lond. Math. Soc.*, Vol. xxx., top of p. 74.

† The proof is identical with that in *Proc. Lond. Math. Soc.*, Vol. xxx., middle of p. 74.

If $a = 1$, the lemma is proved. Suppose then that $a \neq 1$. If

$$S_1 = \tau \equiv C_0 Y_0 T_{2,-1} T_{3,-1} \dots T_{m,-1},$$

which changes the signs of ξ_i, η_i $(i = 0, 2, \dots, m)$, the value $a = -1$ is excluded, since $S_1 \neq T$. Then $S^2 = T_{1,a^2} \neq I$. If $S_1 = I$, then $S = T_{1,a} \neq I$. In either case, we transform by $P_{12} T_{2,-1}$ and obtain a transformation $\neq I$, which leaves ξ_1 and η_1 fixed.

If S_1 be neither τ nor I, there exists in G_m a transformation Σ_1, affecting the same variables as does S_1, and not commutative with S_1 (compare § 3). Then K contains

$$S^{-1}\Sigma_1^{-1} S \Sigma_1 \equiv S_1^{-1} \Sigma_1^{-1} S_1 \Sigma_1 \neq I,$$

which leaves ξ_1 and η_1 unaltered.

(*b*) Let $\beta_{11} = 0$ and β_{1j}, δ_{1j} $(j = 2, \dots, m)$ be not all zero. By § 6, G_m contains a transformation L which does not alter ξ_1 or η_1, and replaces ξ_2 by

$$\beta_{10}\xi_0 + \delta_{10}\eta_0 + \sum_{j=2}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j),$$

subject to (34). Hence K contains $S_1 \equiv L^{-1} S L$, which replaces ξ_1 by $a\xi_1$, and η_1 by $a^{-1}\eta_1 + \xi_2$. The latter function is invariant under the following transformations of G_m:

$$Q_{3,2,\lambda}, \quad V_{2,3,\lambda}, \quad T_{1,\lambda} T_{2,\lambda^{-1}}.$$

If any one of these, say Σ, is not commutative with S_1, then K contains $S_1^{-1} \Sigma^{-1} S_1 \Sigma$, which leaves ξ_1 and η_1 fixed and is not the identity. But, if S_1 be commutative with both $Q_{3,2,\lambda}$ and $V_{2,3,\lambda}$, it replaces ξ_3 and η_3 by, respectively,*

$$\xi_3' = \delta_{22}\xi_3 - \delta_{23}\xi_2, \quad \eta_3' = \delta_{22}\eta_3 - \beta_{23}\xi_2.$$

If S_1 be also commutative with $A \equiv T_{1,\lambda} T_{2,\lambda^{-1}}$ $(\lambda \neq 0, 1)$, we find, upon equating the functions by which AS_1 and $S_1 A$ replace η_2, that $\beta_{23} = \delta_{23} = 0$. Then the transformed of S_1 by $P_{13} T_{3,-1}$ multiplies ξ_1 and η_1 by the same constant. We proceed as in case (*a*).

(*c*) Let $\beta_{11} \neq 0$, and let β_{1j}, δ_{1j} $(j = 2, \dots, m)$ be not all zero. By an evident transformation within G_m, we may take $\delta_{12} \neq 0$. Transforming S by $T_{2,\delta_{12}} T_{3,\delta_{12}}$, we reach a transformation S' with $\beta_{11} \neq 0$, $\delta_{12} = 1$. Employing (34), we find that

$$D \equiv Z_{0,2,\frac{1}{2}\delta_{10}} Y_{0,2,\frac{1}{2}\beta_{10}} Q_{3,2,-\delta_{13}} V_{2,3,\beta_{13}} \dots Q_{m,2,-\delta_{1m}} V_{2,m,\beta_{1m}}$$

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* *Proc. Lond. Math. Soc.*, Vol. xxx., bottom of p. 75.

does not alter ξ_1, η_1, ξ_2, but replaces η_2 by

$$\beta_{10}\xi_0+\delta_{10}\eta_0+(\beta_{11}\delta_{11}+\beta_{12})\,\xi_2+\eta_2+\sum_{j=3}^{m}(\beta_{1j}\xi_j+\delta_{1j}\eta_j).$$

Then K contains $S_1 = D^{-1}S'D$, which replaces ξ_1 by $a\xi_1$, and η_1 by

$$\beta_{11}\xi_1+\delta_{11}\eta_1-\beta_{11}\delta_{11}\xi_2+\eta_2 \quad (\delta_{11} = a^{-1}).$$

We proceed as in *Proc. Lond. Math. Soc.*, Vol. xxx., bottom of p. 76.

(*d*) Let $\beta_{11}\neq 0$, $\beta_{1j}=\delta_{1j}=0$ $(j=2, \ldots, m)$. Then, by (34), β_{10} and δ_{10} are not both zero, while $\delta_{11} = a^{-1}$. We find that

$$S = Z_{0,\,1,\,\frac{1}{2}a\delta_{10}}\,Y_{0,\,1,\,\frac{1}{2}a\beta_{10}}\,T_{1,\,a}\,S_1,$$

where S_1 leaves ξ_1 and η_1 unaltered, and so does not involve them. Since $T_{1,\,\rho}$ transforms $Y_{0,\,1,\,\lambda}$ into $Y_{0,\,1,\,\lambda\rho^{-1}}$ and transforms $Z_{0,\,1,\,\lambda}$ into $Z_{0,\,1,\,\lambda\rho^{-1}}$, since the inverse of $Y_{0,\,1,\,\lambda}$ is $Y_{0,\,1,\,-\lambda}$, since $Y_{0,\,1,\,\lambda}$ and $Z_{0,\,1,\,\mu}$ are commutative, and since

$$Z_{0,\,1,\,\lambda}\,Z_{0,\,1,\,\mu} = Z_{0,\,1,\,\lambda+\mu},$$

we find that

$$T_{1,\,\rho}^{-1}ST_{1\rho}\,.\,S^{-1} = Z_{0,\,1,\,\tau a\delta_{10}}\,Y_{0,\,1,\,\tau a\beta_{10}} \quad [\tau\equiv \tfrac{1}{2}(\rho^{-1}-1)],$$

which belongs to K, and is not the identity of $\rho\neq 1$. Its transformed by $P_{12}T_{2,\,-1}$ leaves ξ_1 and η_1 unaltered and is not the identity.

In the proofs of Lemmas I. and II. we assumed the existence of the variables ξ_i, η_i $(i = 0, 1, 2, 3)$ only. Hence, if $m \overline{>} 4$, we may repeat the process and obtain in K a transformation which leaves ξ_1, η_1, ξ_2, η_2 unaltered and is not the identity. After $m-2$ such steps, we reach a transformation which involves only ξ_i, η_i $(i = 0, m-1, m)$. Transforming it by $P_{1\,m-1}P_{1m}$, which belongs to G_m, we obtain a transformation $\neq I$ which involves only ξ_i, η_i $(i = 0, 1, 2)$. We can readily avoid the case in which it multiplies those indices by -1. In view of the structure of G_2 (§ 5), it follows that K is identical with G_m.

The maximal invariant sub-group of G_m contains no transformation other than the identity and T.

On the Solution of Dynamical Problems in terms of Trigonometric Series. By E. T. WHITTAKER. Received and read November 14th, 1901.

1. The solution of a dynamical problem depends on the integration of a system of ordinary differential equations, in which the time is the independent variable. It is therefore always possible to express the motion in terms of infinite series proceeding in ascending powers of the time, and these series can easily be found; in fact, if the differential equations of the dynamical problem be written in the canonical form

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, n),$$

then when H does not involve the time explicitly the solution is given by $2n$ equations of the type

$$\begin{aligned} q_r = a_r + (t-t_0)(a_r, K) &+ \frac{(t-t_0)^2}{2!}((a_r, K), K) \\ &+ \frac{(t-t_0)^3}{3!}(((a_r, K), K), K) + \frac{(t-t_0)^4}{4!}((((a_r, K), K), K), K), \\ &+ \ldots, \end{aligned}$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, t_0$ are the initial values of $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, t$, respectively, and K is the same function of $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n$ that H is of $q_1, q_2, \ldots, q_n, b_1, \ldots, b_n$, and where

$$(f, \phi) = \sum_{r=1}^{n} \left(\frac{\partial f}{\partial a_r} \frac{\partial \phi}{\partial b_r} - \frac{\partial f}{\partial b_r} \frac{\partial \phi}{\partial a_r} \right).$$

This method can without much difficulty be generalized to meet the case in which H involves the time explicitly; and, from the purely theoretical point of view, the solution of any dynamical problem is completely effected by these series together with the aggregate of the series derived from them by the process of analytic continuation.

The unsatisfactory nature of this result is, however, evident when we consider that these expansions in general converge only for very

limited ranges of values of the time, and that the actual execution of the process of analytic continuation is attended with great difficulties. Moreover, the series fail to give what is often most needed, namely, a ready indication of the number and nature of the distinct types of motion which are possible in the problem. They are, for example, of no assistance to the investigator who aims at classifying the different kinds of orbits described in (say) the problem of three bodies, and determining the periodic and other remarkable solutions.

It may be noticed that the necessity for analytic continuation can be avoided by a change of the independent variable, as was shown by Poincaré;* the integral of the problem is then expressed in series proceeding according to ascending powers of the new variable; but this equally fails to overcome the second of the difficulties mentioned above.

A method is given in this paper for the expression of the solution of a dynamical problem in terms of trigonometric series. Each set of series will represent a family of trajectories, the limiting member of the set representing a position of stable equilibrium in the dynamical system. The process adopted may roughly be described as that of working outwards from a position of stable equilibrium; when such a position has been found the equations are transformed by a change of variables, the new variables being such as would be small if the system were merely describing small oscillations about equilibrium; after this the equations are again transformed by a change of variables, the new variables being such as would change but slowly if the system were describing small oscillations; and then the equations are again repeatedly transformed by changes of the variables, the result of each change being the destruction of one term in the Hamiltonian function—the process in this respect resembling that which underlies Delaunay's lunar theory. When all periodic terms of the Hamiltonian function have been destroyed the equations can be integrated, and the final solution of the dynamical problem is presented in the form of trigonometric series.

As an example of the results attained by this process, the problem of the simple pendulum may be considered. The equations of motion of the pendulum are

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

* *Acta Mathematica*, Vol. IV., p. 211 (1884).

where q is the sine of half the angle made by the pendulum with the vertical, and

$$H = \tfrac{1}{8}p^2 - \tfrac{1}{8}p^2q^2 + 2\mu^2q^2.$$

The form of the solution which is the goal of this paper is

$$q = \frac{2\pi}{K} \sum_{s=1}^{\infty} \frac{c^{\frac{1}{2}(2s-1)}}{1-c^{2s-1}} \sin \frac{(2s-1)\,\pi\mu\,(t-t_0)}{2K},$$

where K and t_0 are two arbitrary constants and

$$c = e^{-\pi K'/K},$$

it being understood that K' is the complete elliptic integral complementary to K. This expansion does, in fact, represent the solution when the type of motion is oscillatory, *i.e.*, the pendulum does not make complete revolutions.

2. Consider then a dynamical problem, expressed by the differential equations

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \quad (r = 1, 2, \ldots, n),$$

where $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$ are the generalized coordinates and momenta, and H is a given function of these quantities, not involving the time t explicitly.

The algebraic solution of the $2n$ simultaneous equations

$$\frac{\partial H}{\partial p_r} = 0, \quad \frac{\partial H}{\partial q_r} = 0 \quad (r = 1, 2, \ldots, n)$$

will furnish in general one or more sets of values $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n$ for the quantities $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$, respectively; and each of these sets of values will correspond to a form of equilibrium or steady motion in the dynamical problem.

Take one of these sets of values $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n$: it is required to find expressions which represent the solution of the dynamical problem when the motion is of a type terminated by this form of equilibrium or steady motion. Thus, if the dynamical problem considered were that of the simple pendulum, and the form of equilibrium chosen were that in which the pendulum hangs vertically downwards at rest, our aim would be to find expressions which represent the solution of the pendulum problem when the motion is of the oscillatory type, *i.e.*, when the pendulum does not make complete revolutions in the vertical plane.

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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

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Take then new variables $q_1', q_2', \ldots, q_n'; p_1', \ldots, p_n'$ defined by the equations

$$q_r = a_r + q_r', \qquad p_r = b_r + p_r' \qquad (r = 1, 2, \ldots, n).$$

The equations become

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \qquad (r = 1, 2, \ldots, n),$$

where H is supposed to be expressed in terms of the new variables.

For sufficiently small values of the variables $q_1', q_2', \ldots, p_n'$, the Hamiltonian function H can be expanded in a multiple power series in the form

$$H = H_0 + H_1 + H_2 + H_3 + \ldots,$$

where H_k denotes terms homogeneous of the k-th degree in $q_1', q_2', \ldots, p_n'$, combined.

Now, since H_0 does not contain any of these variables, it exercises no influence on the differential equations, and so it can be omitted altogether. Moreover, the fact that the differential equations are satisfied when $q_1', q_2', \ldots, p_n'$ are put permanently equal to zero requires that H_1 should vanish identically. The expansion of H in ascending powers of the new variables therefore begins with terms involving their squares and products.

Henceforth we shall suppress the accents, there being no risk of confusion with the old variables, and so the system of differential equations of the problem can be written

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, n),$$

where for sufficiently small values of the variables H can be expanded as

$$H = H_2 + H_3 + H_4 + \ldots,$$

and H_2 can be written in the form

$$H_2 = \tfrac{1}{2}\Sigma\,(a_{rr}q_r^2 + 2a_{rs}q_rq_s) + 2\Sigma b_{rs}q_rp_s + \tfrac{1}{2}\Sigma\,(c_{rr}p_r^2 + 2c_{rs}p_rp_s),$$

where

$$a_{rs} = a_{sr}, \quad c_{rs} = c_{sr}, \quad b_{rs} \neq b_{sr}.$$

3. The next step is to find a new set of variables which will express H_2 in a simpler form.*

* In obtaining the transformation of this article I use a method suggested to me by Mr. T. J. I'A. Bromwich, M.A., Fellow of St. John's College, Cambridge, which seems to furnish the transformation more directly than the method I had myself devised.

For this purpose, consider the set of $2n$ equations

$$\left.\begin{aligned} -\lambda y_r &= a_{r1}x_1 + a_{r2}x_2 + \dots + a_{rn}x_n + b_{r1}y_1 + \dots + b_{rn}y_n \\ \lambda x_r &= b_{1r}x_1 + b_{2r}x_2 + \dots + b_{nr}x_n + c_{r1}y_1 + \dots + c_{rn}y_n \end{aligned}\right\} \quad (r = 1, 2, \dots, n).$$

On solving these, we obtain for λ the equation

$$0 = \begin{vmatrix} a_{11}, & \dots, & a_{1n}; & b_{11}+\lambda, & \dots, & b_{1n} \\ \vdots & & & & & \\ a_{1n}, & \dots, & a_{nn}; & b_{n1}, & \dots, & b_{nn}+\lambda \\ b_{11}-\lambda, & \dots, & b_{n1}; & c_{11}, & \dots, & c_{1n} \\ \vdots & & & & & \\ b_{1n}, & \dots, & b_{nn}-\lambda; & c_{1n}, & \dots, & c_{nn} \end{vmatrix}.$$

Clearly, if λ be a root of this equation, then $-\lambda$ is also a root.

Corresponding to each root λ thus furnished, there will be a set of values for the ratios of the quantities $x_1, x_2, \dots, y_1, \dots, y_n$. Let the roots of the determinantal equation be $\lambda_1, \lambda_2, \dots, \lambda_n, -\lambda_1, -\lambda_2, \dots, -\lambda_n$; let a set of values of $x_1, x_2, \dots, y_n$ corresponding to λ_r be denoted by ${}_rx_1, \dots, {}_rx_n, {}_ry_1, \dots, {}_ry_n$; and let a set corresponding to the root $-\lambda_r$ be denoted by ${}_{-r}x_1, \dots, {}_{-r}x_n, {}_{-r}y_1, \dots, {}_{-r}y_n$. Then we have

$$-\lambda_r\,{}_ry_p = a_{p1}\,{}_rx_1 + \dots + a_{pn}\,{}_rx_n + b_{p1}\,{}_ry_1 + \dots + b_{pn}\,{}_ry_n.$$

$$\lambda_r\,{}_rx_p = b_{1p}\,{}_rx_1 + \dots + b_{np}\,{}_rx_n + c_{p1}\,{}_ry_1 + \dots + c_{pn}\,{}_ry_n.$$

Multiply by ${}_sx_p$ and ${}_sy_p$, and add to the similar results for other suffixes. Then

$$\lambda_r \Sigma ({}_rx\,{}_sy - {}_sx\,{}_ry) = H(r, s),$$

where the summation is extended over all *equal* suffixes of x, y, and where

$$H(r, s) = a_{11}\,{}_rx_1\,{}_sx_1 + a_{12}({}_rx_1\,{}_sx_2 + {}_sx_1\,{}_rx_2) + \dots + b_{11}({}_rx_1\,{}_sy_1 + {}_sx_1\,{}_ry_1) + \dots$$
$$\dots + c_{11}\,{}_ry_1\,{}_sy_1 + \dots,$$

which is symmetrically related to r and s.

Interchanging r and s, we have

$$\lambda_s \Sigma ({}_sx\,{}_ry - {}_rx\,{}_sy) = H(r, s).$$

Thus $$(\lambda_r + \lambda_s) \Sigma ({}_rx\,{}_sy - {}_sx\,{}_ry) = 0.$$

So, unless $\lambda_r + \lambda_s$ is zero, we have

$$\Sigma ({}_rx\,{}_sy - {}_sx\,{}_ry) = 0,$$

and consequently $$H(r, s) = 0.$$

If $$\lambda_r + \lambda_s = 0,$$

we have $${}_s x = {}_r x, \quad {}_s y = {}_{-r} y;$$

and therefore $$\lambda_r \Sigma ({}_r x\, {}_{-r} y - {}_{-r} x\, {}_r y) = H(r, -r).$$

If then we make the substitutions expressed by the equations

$$q_r = {}_1 x_r\, q_1' + {}_2 x_r\, q_2' + \ldots + {}_n x_r\, q_n' + {}_{-1} x_r\, p_1' + \ldots + {}_{-n} x_r\, p_n', \qquad (r = 1, 2, \ldots, n),$$

and $p =$ a similar expression with y's instead of x's, it is clear that the coefficient of $\delta q_r' \Delta p_s'$ in $\Sigma (\delta q_1 \Delta p_1 - \Delta q_1 \delta p_1)$, where δ and Δ denote any two independent modes of variation, is $\Sigma ({}_r x\, {}_s y - {}_s x\, {}_r y)$, which is zero when $\lambda_r + \lambda_s$ is not zero. Thus $\Sigma (\delta q_1 \Delta p_1 - \Delta q_1 \delta p_1)$ contains no terms except such as $\Sigma (\delta q_r' \Delta p_r' - \Delta q_r' \delta p_r')$, and the coefficient of this term is $\Sigma ({}_r x\, {}_{-r} y - {}_{-r} x\, {}_r y)$. Now hitherto the actual values of ${}_r x$, ${}_r y$ have not been fixed, as only their ratios are determined from their equations of definition. We can therefore choose their values so that

$$\Sigma ({}_r x\, {}_{-r} y - {}_{-r} x\, {}_r y) = 1$$

for each value of r; and then we have

$$\Sigma (\delta q_r \Delta p_r - \Delta q_r \delta p_r) = \Sigma (\delta q_r' \Delta p_r' - \Delta q_r' \delta p_r').$$

Now this last equation expresses the condition that a transformation from the variables $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$, to the variables $q_1', q_2', \ldots, q_n'$, $p_1', \ldots, p_n'$, shall be *canonical*, *i.e.*, that it shall leave unaltered the Hamiltonian form of any system of Hamiltonian differential equations in which these quantities are the variables. Further, if in H_2 we substitute for the old variables $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$, in terms of the new variables $q_1', q_2', \ldots, q_n', p_1', \ldots, p_n'$, we obtain

$$H_2 = \sum_{r=1}^{n} H(r, -r)\, q_r' p_r',$$

or $$H_2 = \sum_{r=1}^{n} \lambda_r q_r' p_r'.$$

Thus, when this change of variables is made in the differential equations of our dynamical problem, as given at the end of § 2, the differential equations take the form

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial p_r'} \qquad (r = 1, 2, \ldots, n),$$

where for sufficiently small values of the variables H can be expanded as

$$H = H_2 + H_3 + H_4 + \ldots,$$

in which H_k is homogeneous of degree k in $q_1', q_2', \ldots, q_n', p_1', \ldots, p_n'$, and H_2 can be written in the form

$$H_2 = \sum_{r=1}^{n} \lambda_r q_r' p_r'.$$

4. To this system apply the further transformation from the variables $q_1', q_2', \ldots, q_n', p_1', \ldots, p_n$ to variables $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$, defined by the equations

$$q_r = \frac{\partial W}{\partial p_r}, \qquad p_r' = \frac{\partial W}{\partial q_r'} \qquad (r = 1, 2, \ldots, n),$$

where
$$W = \sum_{r=1}^{n} p_r q_r' - \tfrac{1}{2} \sum_{r=1}^{n} \frac{p_r^2}{\lambda_r} - \tfrac{1}{4} \sum_{r=1}^{n} \lambda_r q_r'^2.$$

From the form of this transformation, it is known to be canonical, and so the differential equations retain their Hamiltonian form. Moreover, since the transformation is linear, the quantities $H_2, H_3, \ldots$ will still be homogeneous polynomials in the new variables $q_1, q_1, \ldots, q_n, p_1, \ldots, p_n$; and in particular it is easily seen that

$$H_2 = \tfrac{1}{2} \sum_{r=1}^{n} (p_r^2 - \lambda_r^2 q_r^2).$$

The quantities λ_r, which have been obtained as the roots of a determinant, are constants, depending on the constitution of the dynamical system whose motion we are considering; in a very large class of cases, which for practical purposes is the most important class, they are purely imaginary, so that the quantities $-\lambda_r^2$ are real and positive; in this case the particular solution of the dynamical problem from which we start, and which is to be the limiting case of the family of solutions we propose to find, may be called a position of stable equilibrium or steady motion. We shall confine our attention to systems of this kind, and to indicate this we shall write λ_r^2 for $-\lambda_r^2$. Thus (summarizing the last three sections of this paper) *the equations of motion of the dynamical system have been brought to the form*

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, n),$$

where
$$H = H_2 + H_3 + H_4 + \ldots,$$

in which H_r is a homogeneous polynomial in the variables $q_1, q_2, \ldots, q_n$, $p_1, \ldots, p_n$; and, in particular,

$$H_2 = \tfrac{1}{2} \sum_{r=1}^{n} (p_r^2 + \lambda_r^2 q_r^2),$$

where the quantities λ_r are supposed real.

It is clear that, in order to consider the small oscillations of the system about the position which has been the starting-point of our work, we should merely have to neglect altogether the part $H_3 + H_4 + H_5 + \ldots$ of H, and that the variables $q_1, q_2, \ldots, q_n$ would then be the "principal coordinates" for the small oscillations.

5. The system of differential equations which has been obtained will now be further transformed by applying to it a transformation from the variables $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$, to new variables $q_1', q_2', \ldots, q_n'$, $p_1', \ldots, p_n'$, defined by the system of equations

$$p_r' = \frac{\partial W}{\partial q_r'}, \qquad q_r = \frac{\partial W}{\partial p_r} \qquad (r = 1, 2, \ldots, n),$$

where $$W = \sum_{r=1}^{n} \left[q_r' \sin^{-1} \frac{p_r}{(2\lambda_r q_r')^{\frac{1}{2}}} + \frac{p_r}{2\lambda_r} \{2\lambda_r q_r' - p_r^2\}^{\frac{1}{2}} \right].$$

From the form in which this transformation is expressed, it is clearly canonical, and so will leave unaltered the Hamiltonian form of the differential equations.

The equations connecting the old and new variables are

$$\left.\begin{aligned} p_r &= (2\lambda_r q_r')^{\frac{1}{2}} \sin p_r' \\ q_r &= (2q_r')^{\frac{1}{2}} \lambda_r^{-\frac{1}{2}} \cos p_r' \end{aligned}\right\} \quad (r = 1, 2, \ldots, n).$$

The differential equations now become

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \qquad (r = 1, 2, \ldots, n),$$

where $$H = \lambda_1 q_1' + \lambda_2 q_2' + \ldots + \lambda_n q_n' + H_3 + H_4 + \ldots,$$

and now H_r denotes an aggregate of terms which are homogeneous of degree $\frac{r}{2}$ in the quantities q_r', and homogeneous of degree r in the quantities $\cos p_r'$, $\sin p_r'$.

Since a product of powers of the quantities $\cos p_r'$, $\sin p_r'$ can be expressed as a sum of sines and cosines of angles of the form

$$n_1 p_1' + n_2 p_2' + \ldots + n_n p_n',$$

where $n_1, n_2, \ldots, n_n$ have integer or zero values, it follows that H_r can be expressed as the sum of a finite number of terms, each of the form

$$q_1'^{m_1} q_2'^{m_2} \ldots q_n'^{m_n} \begin{matrix}\sin \\ \cos\end{matrix} (n_1 p_1' + n_2 p_2' + \ldots + n_n p_n'),$$

where $$m_1 + m_2 + \ldots + m_n = \frac{r}{2}$$

and $$|n_1| + |n_2| + \ldots + |n_n| \leqslant r,$$

since we must have $$|n_r| \leqslant 2m_r \quad (r = 1, 2, \ldots, n).$$

Let all the quantities H_r be transformed in this way, so that H is expressed in the form

$$H = \Sigma A^{m_1, m_2, \ldots, m_n}_{n_1, n_2, \ldots, n_n} q_1'^{m_1} q_2'^{m_2} \ldots q_n'^{m_n} \begin{matrix}\sin \\ \cos\end{matrix} (n_1 p_1' + n_2 p_2' + \ldots + n_n p_n'),$$

where for each term we have

$$|n_1| + |n_2| + \ldots + |n_n| \leqslant 2(m_1 + m_2 + \ldots + m_n),$$

and the series is clearly absolutely convergent for all values of $p_1', \ldots, p_n'$, provided $q_1', q_2', \ldots, q_n'$ do not exceed certain limits of magnitude. From the absolute convergence it follows that the order of the terms can be rearranged in any arbitrary way; we shall suppose them so ordered that all the terms involving the same argument $n_1 p_1' + \ldots + n_n p_n'$ are collected together, and thus H can be expressed in the form

$$H = a_{0,0,0,\ldots,0} + \Sigma a_{n_1, n_2, \ldots, n_n} \cos (n_1 p_1' + \ldots + n_n p_n')$$
$$+ \Sigma b_{n_1, n_2, \ldots, n_n} \sin (n_1 p_1' + \ldots + n_n p_n'),$$

where the quantities a and b are functions of $q_1', q_2', \ldots, q_n'$, and the expansion of $a_{n_1, n_2, \ldots, n_n}$ or $b_{n_1, n_2, \ldots, n_n}$ in powers of $q_1', \ldots, q_n'$ contains no terms of order lower than $\frac{1}{2}\{|n_1| + |n_2| + \ldots + |n_n|\}$; and where the summations extend over all positive and negative integer and zero values of $n_1, n_2, \ldots, n_n$, except the combination

$$n_1 = n_2 = \ldots = n_n = 0.$$

Moreover, the expansion of $a_{0,0,\ldots,0}$ begins with the terms

$$\lambda_1 q_1' + \lambda_2 q_2' + \ldots + \lambda_n q_n';$$

and, when $q_1', q_2', \ldots, q_n'$ are small, these are the most important terms

in H, since they contribute terms independent of $q_1', q_2', \ldots, q_n'$ to the differential equations.

For convenience we shall often speak of $q_1', q_2', \ldots, q_n'$ as "small," in order to have a definite idea of the relative importance of the terms which occur. It will be understood that $q_1', q_2', \ldots, q_n'$ are not, however, infinitesimal, and, in fact, are not restricted at all in magnitude except so far as is required to ensure the convergence of the various series in which they occur.

To avoid unnecessary complexity, we shall ignore the terms

$$\Sigma b_{n_1, n_2, \ldots, n_n} \sin(n_1 p_1' + \ldots + n_n p_n')$$

in H, as they are to be treated in the same way as the terms

$$\Sigma a_{n_1, n_2, \ldots, n_n} \cos(n_1 p_1' + \ldots + n_n p_n'),$$

and their presence complicates, but does not in any important respect modify, the later developments.

The form to which the problem has now been brought may therefore be stated as follows (suppressing the accents in the new variables, as there is no longer any risk of confusion with the old variables).

The equations of motion are

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, n),$$

where $H = a_{0,0,\ldots,0} + \Sigma a_{n_1, n_2, \ldots, n_n} \cos(n_1 p_1 + n_2 p_2 + \ldots + n_n p_n)$,

and the quantities a are functions of $q_1, q_2, q_3, \ldots, q_n$ *only; moreover, the periodic part of H is small compared with the non-periodic part* $a_{0,0,\ldots,0}$; *a term which has for argument* $n_1 p_1 + n_2 p_2 + \ldots + n_n p_n$ *has its coefficient* $a_{n_1, n_2, \ldots, n_n}$ *at least of the order* $\frac{1}{2}\{|n_1| + |n_2| + \ldots + |n_n|\}$ *in the small quantities* $q_1, q_2, \ldots, q_n$; *and the expansion of* $a_{0,0,\ldots,0}$ *begins with terms* $\lambda_1 q_1 + \lambda_2 q_2 + \ldots + \lambda_n q_n$.

It follows from this that when the variables $q_1, q_2, \ldots, q_n$ are small they are nearly constant, while the variables $p_1, p_2, \ldots, p_n$ vary almost proportionally to the time.

6. To the variables of this system of differential equations we shall now apply a further transformation, the effect of which will be the removal of one of the periodic terms in H; the aim being to still further accentuate the feature already noted, namely, that the

non-periodic part of H is much more important than the periodic part.*

Let then one of the periodic terms in H be selected—say

$$a_{n_1, n_2, \ldots, n_n} \cos(n_1 p_1 + n_2 p_2 + \ldots + n_n p_n).$$

Write $H = a_{0, 0, \ldots, 0} + a_{n_1, n_2, \ldots, n_n} \cos(n_1 p_1 + n_2 p_2 + \ldots + n_n p_n) + R,$

so that R denotes the rest of the periodic terms of H. When we wish to put in evidence the fact that $a_{n_1, n_2, \ldots, n_n}$ is a function of its arguments, we shall write it $a_{n_1, n_2, \ldots, n_n}(q_1, q_2, \ldots, q_n)$.

Now apply to the variables the transformation from $q_1, q_2, \ldots, q_n$; $p_1, \ldots, p_n$ to $q_1', q_2', \ldots, q_n'$; $p_1', \ldots, p_n'$, defined by the equations

$$p_r' = \frac{\partial W}{\partial q_r'}, \qquad q_r = \frac{\partial W}{\partial p_r} \qquad (r = 1, 2, \ldots, n),$$

where $\quad W = q_1' p_1 + q_2' p_2 + \ldots + q_n' p_n + f(q_1', q_2', \ldots, q_n', \theta)$

and $\quad \theta = n_1 p_1 + n_2 p_2 + \ldots + n_n p_n.$

We shall suppose that f is a function, as yet undetermined, of the arguments indicated. The transformation is seen at once from its form to be canonical; so the Hamiltonian form of the differential equations will not be affected by it, and the problem is expressed by the system

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \qquad (r = 1, 2, \ldots, n),$$

where
$$H = a_{0, 0, \ldots, 0}\left(q_1' + n_1 \frac{\partial f}{\partial \theta}, \ldots, q_n' + n_n \frac{\partial f}{\partial \theta}\right)$$
$$+ a_{n_1, n_2, \ldots, n_n}\left(q_1' + n_1 \frac{\partial f}{\partial \theta}, \ldots, q_n' + n_n \frac{\partial f}{\partial \theta}\right) \cos\theta + R,$$

and θ and R are supposed to be expressed in terms of the new variables by means of the equations of transformation

$$p_r' = p_r + \frac{\partial f}{\partial q_r'}, \quad q_r = q_r' + n_r \frac{\partial f}{\partial \theta} \quad (r = 1, 2, \ldots, n).$$

The function f is, as yet, undetermined and at our disposal. It will

* The analogy of this with the method of Delaunay's lunar theory will be noticed. Although the analysis is different from Delaunay's, the idea is essentially the same.

be chosen so as to satisfy the condition that θ shall identically disappear from the expression

$$a_{0,0,\ldots,0}\left(q_1'+n_1\frac{\partial f}{\partial\theta},\ \ldots,\ q_n'+n_n\frac{\partial f}{\partial\theta}\right)$$
$$+a_{n_1,n_2,\ldots,n_n}\left(q_1'+n_1\frac{\partial f}{\partial\theta},\ \ldots,\ q_n'+n_n\frac{\partial f}{\partial\theta}\right)\cos\theta;$$

so that this quantity is a function of q_1', q_2', ..., q_n' alone. Put it equal to $a'_{0,0,\ldots,0}$, where $a'_{0,0,\ldots,0}$ is a function of q_1', q_2', ..., q_n', as yet undetermined. Then the equation

$$a_{0,0,\ldots,0}\left(q_1'+n_1\frac{\partial f}{\partial\theta},\ \ldots,\ q_n'+n_n\frac{\partial f}{\partial\theta}\right)$$
$$+a_{n_1,n_2,\ldots,n_n}\left(q_1'+n_1\frac{\partial f}{\partial\theta},\ \ldots,\ q_n'+n_n\frac{\partial f}{\partial\theta}\right)\cos\theta=a'_{0,0,\ldots,0}.$$

determines $\dfrac{\partial f}{\partial\theta}$ in terms of q_1', q_2', ..., q_n', $a'_{0,0,\ldots,0}$ and $\cos\theta$.

Suppose the solution of this equation for $\dfrac{\partial f}{\partial\theta}$ is expressed in the form of a series of cosines of multiples of θ (which can be done, for instance, by successive approximation), so that

$$\frac{\partial f}{\partial\theta}=c_0+\sum_{k=1}^{\infty}c_k\cos k\theta,$$

where c_0, c_1, c_2, ... are known functions of q_1', q_2', ..., q_n', $a'_{0,0,\ldots,0}$.

Now $a'_{0,0,\ldots,0}$ is as yet undetermined, and is at our disposal. Impose the condition that c_0 is to be zero. This determines $a'_{0,0,\ldots,0}$ as a function of q_1', q_2', ..., q_n'; and, on substituting its value in the series for $\dfrac{\partial f}{\partial\theta}$, we have

$$\frac{\partial f}{\partial\theta}=\sum_{k=1}^{\infty}c_k\cos k\theta,$$

where now c_1, c_2, c_3, ... are known functions of q_1', q_2', ..., q_n'.

Integrating this equation with respect to θ, and for our purpose taking the constant of integration to be zero, we have

$$f=\sum_{k=1}^{\infty}\frac{c_k}{k}\sin k\theta.$$

The equations defining the transformation now become

$$\left.\begin{aligned} p'_r &= p_r + \sum_{k=1}^{\infty} \frac{1}{k} \frac{\partial c_k}{\partial q'_r} \sin k\theta \\ q_r &= q'_r + n_r \sum_{k=1}^{\infty} c_k \cos k\theta \end{aligned}\right\} \quad (r = 1, 2, \ldots, n).$$

Multiply the first set of these equations by $n_1, n_2, \ldots, n_n$, respectively, and add them; writing

$$n_1 p'_1 + n_2 p'_2 + \ldots + n_n p'_n = \theta',$$

we have $\theta' = \theta + \sum_{k=1}^{\infty} \frac{1}{k} \left(n_1 \frac{\partial c_k}{\partial q'_1} + n_2 \frac{\partial c_k}{\partial q'_2} + \ldots + n_n \frac{\partial c_k}{\partial q'_n} \right) \sin k\theta.$

Reversing this series, we have

$$\theta = \theta' + \sum_{k=1}^{\infty} d_k \sin k\theta',$$

where $d_1, d_2, \ldots$ are known functions of $q'_1, q'_2, \ldots, q'_n$. Substituting this value of θ in the equations of transformation, they become

$$\left.\begin{aligned} p_r &= p'_r + \sum_{k=1}^{\infty} {}_r e_k \sin k\theta' \\ q_r &= q'_r + n_r \sum_{k=1}^{\infty} g_k \cos k\theta \end{aligned}\right\} \quad (r = 1, 2, \ldots, n),$$

where all the quantities ${}_r e_k$, g_k are known functions of $q'_1, q'_2, \ldots, q'_n$.

Now, before the transformation, the quantity R consisted of an aggregate of terms of the type

$$R = \Sigma a_{m_1, m_2, \ldots, m_n} \cos (m_1 p_1 + \ldots + m_n p_n).$$

When the values just found for $q_1, \ldots, q_n, p_1, \ldots, p_n$ are substituted in this expression, and the series is reduced by replacing powers and products of trigonometrical functions of $p'_1, p'_2, \ldots, p'_n$ by cosines of multiples of $p'_1, p'_2, \ldots, p'_n$, it is clear that R will consist of an aggregate of terms of the type

$$R = \Sigma a'_{m_1, m_2, \ldots, m_n} \cos (m_1 p'_1 + m_2 p'_2 + \ldots + m_n p'_n),$$

where the quantities a' are known functions of $q'_1, q'_2, \ldots, q'_n$.

We see therefore, omitting the accents of the new variables, that,

after the transformation has been effected, the dynamical problem is still expressed by a system of equations of the form

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \dots, n),$$

where $\quad H = a_{0,\,0,\,\dots,\,0} + \Sigma\, a_{m_1,\,m_2,\,\dots,\,m_n} \cos(m_1 p_1 + \dots + m_n p_n),$

and where the coefficients a *are known functions of* $q_1, q_2, \dots, q_n$.

7. Let us now review the whole effect of the transformation described in the last article. The differential equations of motion have the same general form as before; but one term, namely,

$$a_{n_1,\,n_2,\,\dots,\,n_n} \cos(n_1 p_1 + \dots + n_n p_n),$$

has been transferred from the periodic part of H to its non-periodic part; the periodic part of H is less important, in comparison with the non-periodic part, than it was before the transformation was made.

8. Having now completed the absorption of this periodic term into the non-periodic part of H, we proceed to absorb one of the periodic terms of the new expansion of H into the non-periodic part, by a repetition of the same process. In this way we can continually enrich the non-periodic part of H at the expense of the periodic part, and ultimately, after a number of applications of the transformation, the periodic part of H will become so insignificant that it may be neglected. Let $\alpha_1, \alpha_2, \dots, \alpha_n$; $\beta_1, \beta_2, \dots, \beta_n$ be the variables at which we arrive as a result of the final transformation. Then the equations of motion are

$$\frac{d\alpha_r}{dt} = \frac{\partial H}{\partial \beta_r}, \qquad \frac{d\beta_r}{dt} = -\frac{\partial H}{\partial \alpha_r} \qquad (r = 1, 2, \dots, n),$$

where H, consisting only of its non-periodic part, is a function of $\alpha_1, \alpha_2, \dots, \alpha_n$ only. We have, therefore,

$$\frac{d\alpha_r}{dt} = 0, \qquad \beta_r = -\int \frac{\partial H}{\partial \alpha_r}\, dt,$$

and so the quantities α_r are constants, while the quantities β_r are of the form

$$\beta_r = \mu_r t + \epsilon_r \qquad (r = 1, 2, \dots, n),$$

where $\quad \mu_r = -\dfrac{\partial H}{\partial \alpha_r};$

the quantities ϵ_r are arbitrary constants, and the part of μ_r independent of $a_1, a_2, \ldots, a_n$ is $-\lambda_r$.

9. Having now solved the equations of motion in their final form, it remains only to express the original coordinates of the dynamical problem in terms of the ultimate coordinates $a_1, a_2, \ldots, \beta_n$. Remembering that the product of any number of canonical transformations is a canonical transformation, it is easily seen that the variables $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$ used at the end of § 5 can be expressed in terms of $a_1, a_2, \ldots, a_n, \beta_1, \ldots, \beta_n$ by equations of the form

$$\left.\begin{aligned} \beta_r &= p_r + \Sigma \frac{\partial k_{m_1, m_2, \ldots, m_n}}{\partial a_r} \sin(m_1 p_1 + \ldots + m_n p_n) \\ q_r &= a_r + \Sigma m_r k_{m_1, m_2, \ldots, m_n} \cos(m_1 p_1 + \ldots + m_n p_n) \end{aligned}\right\} \quad (r = 1, 2, \ldots, n),$$

or of the form

$$\left.\begin{aligned} q_r &= f_r(a_1, a_2, \ldots, a_n) + \Sigma a_{m_1, m_2, \ldots, m_n} \cos(m_1\beta_1 + \ldots + m_n\beta_n) \\ p_r &= \beta_r + \Sigma_r b_{m_1, m_2, \ldots, m_n} \sin(m_1\beta_1 + \ldots + m_n\beta_n) \end{aligned}\right\} \quad (r = 1, 2, \ldots, n),$$

where the quantities a and b are functions of $a_1, a_2, \ldots, a_n$.

From this it follows that the variables $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$ of § 1, in terms of which the dynamical problem was originally expressed, are obtained by the process of this paper in the form of trigonometric series, proceeding in sines and cosines of sums of multiples of the n angles $\beta_1, \beta_2, \ldots, \beta_n$. These angles are linear functions of the time, of the form $\mu_r t + \epsilon_r$; the quantities ϵ_r are n of the $2n$ arbitrary constants of the solution, while the quantities μ_r are of the form

$$\mu_r = \lambda_r + \sum_{k_1, k_2, \ldots} a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n}.$$

The coefficients in the trigonometric series are functions of the arbitrary constants $a_1, a_2, \ldots, a_n$ only.

The expansions thus found represent a family of solutions of the dynamical problem, the limiting member of the family being the position of equilibrium or steady motion which was our starting point.

If we were to consider those solutions which differ only slightly from this terminal case, we might neglect all powers of $a_1^{\frac{1}{2}}, a_2^{\frac{1}{2}}, \ldots, a_n^{\frac{1}{2}}$

above the first, and then it is easily seen that the quantities $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ would reduce to $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, and so the expansions would become the well-known formulæ which represent small oscillations of the dynamical system about the position of equilibrium or steady motion.

In practice, the expansions can be carried as far as may be desired by including the requisite powers of $q_1, q_2, \ldots, q_n$, and so of $a_1, a_2, \ldots, a_n$. As a simple example of this, it will be found by this method that the dynamical system expressed by the differential equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

where

$$H = \tfrac{1}{2}p^2 + \frac{l^4k^2}{2q^2} - \frac{l^3k^2}{q}$$

possesses a family of solutions represented by the expansion (retaining only terms of order less than $a^{\frac{3}{2}}$)

$$q = l + \frac{3a}{kl} + \left(\frac{2a}{k}\right)^{\frac{1}{2}} \cos\beta - \frac{3a}{2kl}\cos 2\beta,$$

where

$$\beta = -\left(k + \frac{aa}{2l^3}\right)t + \epsilon,$$

and a and ϵ are arbitrary constants.

This family of solutions is terminated by the equilibrium-solution

$$q = l,$$

which corresponds to the value zero of a. The small oscillations of the system about this equilibrium-position would be derived by neglecting all powers of a above $a^{\frac{1}{2}}$ in the above expansion, which gives

$$q = l + \left(\frac{2a}{k}\right)^{\frac{1}{2}} \cos(-kt + \epsilon),$$

where a and ϵ are the arbitrary constants; a result in accord with the customary form.

Thursday, December 12th, 1901.

Major MACMAHON, R.A., F.R.S., Vice-President, in the Chair.

Ten members present.

The Chairman moved the adoption of the Treasurer's report, after reading the Auditor's report, and Prof. Alfred Lodge seconded the motion, as well as votes of thanks to the Treasurer and the Auditor. The votes were carried unanimously.

Messrs. G. Birtwistle, B.A., Fellow of Pembroke College, Cambridge; and Augustus P. Thompson, B.A., Scholar of Pembroke College, Cambridge; and the Rev. J. Cullen, S.J., were elected members.

Mr. R. J. Dallas was admitted into the Society.

Prof. Love communicated a paper by Mr. J. H. Michell, "On the Flexure of a Circular Plate." Prof. Lamb spoke on the subject of the paper.

The following presents were made to the Library :—

"Educational Times," December, 1901.

"Indian Engineering," Vol. xxx., Nos. 17–20, Oct. 26–Nov. 16, 1901.

"Mathematical Gazette," Vol. II., No. 30, 1901.

"Kansas University Quarterly," Vol. II., No. 6 ; April, 1901.

"Supplemento al Periodico di Matematica," Anno V., Fasc. 1, Nov., 1901 ; Livorno.

Frick, J.—"On Liquid Air and its Application (Liquid Air Wells)," 8vo ; London, 1901.

The following exchanges were received :—

"Proceedings of the Royal Society," Vol. LXIX., No. 452, 1901.

"Bulletin of the American Mathematical Society," Series 2, Vol. VIII., No. 2 ; New York, 1901.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XIV., No. 5 ; Coimbra, 1901.

"Bulletin des Sciences Mathématiques," Tome XXV., Oct., 1901 ; Paris.

"Annali di Matematica," Série 3, Tomo VI., Fasc. 4 ; Milano, 1901.

"Archives Néerlandaises," Serie 2, Tome VI. ; La Haye, 1901.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. X., Fasc. 9 ; Roma, 1901.

"Annales de la Faculté des Sciences de Marseille," Tome XI., Fasc. 1–9 ; Paris, 1901.

"Proceedings of the Royal Irish Academy," Vol. VI., Nos. 2, 3, Jan., Oct., 1901 ; Dublin.

The Flexure of a Circular Plate. *By* J. H. Michell.

Received November 24th, 1901. Read December 12th, 1901.

1. The flexure of a thin circular plate, clamped along its edge and loaded in any manner, has been discussed by Clebsch in his *Theorie der Elasticität.** He obtains expressions in series for the deflection due to a concentrated load at any point of the plate and remarks on the complexity of the result. The process of inversion, described in a previous paper,† leads to the solution for an eccentric load by the inversion of that for a central load, and thus reduces the former to the simplicity of the latter. Mathematically, the problem is nothing more than the expression of "Green's function" for the differential equation $\nabla^4 w = 0$ within a circular boundary. The solution of the similar problem of the two-dimensional flow of viscous liquid within a circular boundary is merely noted. Rayleigh‡ has already treated this problem in another manner.

2. The deflection w of the portions of a plate§ free from load satisfies the differential equation $\nabla^4 w = 0$, and the conditions at the clamped edge are $w = 0$, $\partial w/\partial n = 0$, δn being an element of normal to the edge. Inverting with respect to the origin O of the polar coordinates (r, θ), we obtain a solution $w' = w/r^2$ in the inverse plane, satisfying $\nabla^4 w' = 0$ in that plane and also, clearly, satisfying the conditions $w' = 0$, $\partial w'/\partial n' = 0$ for a clamped edge, along the inverse of the clamped edge in the original plane.

In the neighbourhood of a concentrated load W, w takes the form $\kappa W r_1^2 \log r_1 + w_1$, where r_1 is the distance from the point of application C of the load, κ depends on the stiffness of the plate, and w_1 is regular. This need not be formally proved, as it follows from the known solution quoted below. Let O' be the inverse of C with respect to O, and let ρ, ρ_1 be the distances from O, O' respectively of the inverse of the point at distances r, r_1 from O, C respectively.

* St. Venant's edition, § 76.

† "The Inversion of Plane Stress," *Proc. Lond. Math. Soc.*, Vol. xxxiv., p. 134.

‡ *Phil. Mag.*, Vol. xxxvi., 1893, p. 354.

§ The plate is supposed isotropic in its plane.

Taking unit radius of inversion and writing $OC = c$, we have

$$\frac{r_1}{\rho_1} = \frac{c}{\rho}, \quad r\rho = 1;$$

and therefore the inverse of the form $\kappa W r_1^2 \log r_1$ is

$$\kappa W c^2 \rho_1^2 \log \rho_1 \frac{c}{\rho} \quad \text{or} \quad \kappa W c^2 \rho_1^2 \log \rho_1 + \kappa W c^2 \rho_1^2 \log \frac{c}{\rho}.$$

Thus there is a concentrated load Wc^2 at the point O'. The fact that O is also a singular point is immaterial, as we shall suppose it outside the plate.

3. The relation between the deflections w, w' at corresponding points P, P' may be written $w/OP = w'/OP'$. Remembering the smallness of w, w', this shows that the deflected position of the point P' is the inverse of that of P, or that the strained form of one plate is the geometrical inverse of that of the other. Now it is known that the lines of curvature of a surface invert into lines of curvature. Hence the lines of curvature or flexure of the two strained plates correspond. The quantitative relations between the stresses in the two plates are readily obtained. The couples on a polar element of a plate can be written

$$K_1 = C\nabla^2 w - C(1-\sigma)\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right),$$

$$K_2 = C\nabla^2 w - C(1-\sigma)\frac{\partial^2 w}{\partial r^2},$$

$$H = C(1-\sigma)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right);$$

and, as in the previous paper, these make the couples on the polar element at the corresponding point in the inverse solution

$$K_1' = C\left\{r^2\nabla^2 w + 4\left(w - r\frac{\partial w}{\partial r}\right)\right\}$$
$$\qquad - C(1-\sigma)\left\{r^2\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + 2\left(w - r\frac{\partial w}{\partial r}\right)\right\},$$

$$K_2' = C\left\{r^2\nabla^2 w + 4\left(w - r\frac{\partial w}{\partial r}\right)\right\} - C(1-\sigma)\left\{r^2\frac{\partial^2 w}{\partial r^2} + 2\left(w - r\frac{\partial w}{\partial r}\right)\right\},$$

$$H' = -C(1-\sigma)r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right),$$

so that
$$K_1' = r^2 K_1 + 2C(1+\sigma)\left(w - r\frac{\partial w}{\partial r}\right),$$
$$K_2' = r^2 K_2 + 2C(1+\sigma)\left(w - r\frac{\partial w}{\partial r}\right),$$
$$H' = -r^2 H.$$

It follows that the resultant couple across a line-element $\delta s'$ of the inverse plate is compounded of (a) a couple equal to that across the corresponding line-element δs in the original plate and in the corresponding direction, (b) a couple of magnitude $2C(1+\sigma)\left(w - r\frac{\partial w}{\partial r}\right)\delta s'$ with its axis along the element. In particular, it follows again that the lines of principal flexure in the two plates correspond.

4. The deflection of a clamped circular plate of radius a due to a central load W is*

$$w = \kappa W\left(\frac{a^2 - r_1^2}{2} - r_1^2 \log\frac{a}{r_1}\right).$$

Invert with respect to the point O outside the plate as in § 2. If a is the radius of the new plate, C' its centre, and $O'C' = h$, then $a/c = h/a$; and therefore

$$w' = \kappa W c^2 \left\{ \tfrac{1}{2}\rho_1^2 \left(\frac{h^2\rho^2}{a^2\rho_1^2} - 1\right) - \rho_1^2 \log\frac{h\rho}{a\rho_1} \right\}.$$

Hence, changing the notation to a more convenient, the deflection w, due to a load W on a clamped circular plate of radius a at a point distant h from the centre, is given by

$$w = \kappa W r^2 \left\{ \tfrac{1}{2}\left(\frac{h^2 r'^2}{a^2 r^2} - 1\right) - \log\frac{hr'}{ar} \right\},$$

where r is the distance from the load and r' that from the inverse point.

If the radius of the circle is made infinite, we obtain the solution for the deflection of an infinite plate clamped along an infinite straight boundary, due to a concentrated load. The solution is

$$w = \kappa W r^2 \left\{ \tfrac{1}{2}\left(\frac{r'^2}{r^2} - 1\right) - \log\frac{r'}{r} \right\},$$

---

* See *e.g.*, St. Venant's *Clebsch*, p. 777.

where the inverse point is now the image of the load in the boundary. With my brother's assistance I have drawn the accompanying figures showing the contour lines (Fig. I.) for a circular plate loaded at a point bisecting a radius (indicated by a cross), (Fig. II.) for an infinite plate loaded at a point similarly indicated. Fig. I. (*a*) shows the deflection along the diameter of the circle on which the load is, Fig. I. (*b*) the deflection along the perpendicular diameter (inner curve), and the profile of the plate viewed from a distant point on the former diameter (outer curve).

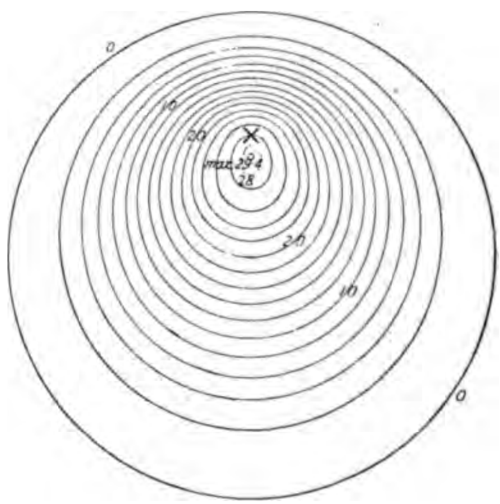


Fig. I.

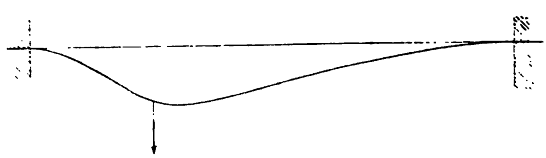


Fig. I. (*a*).

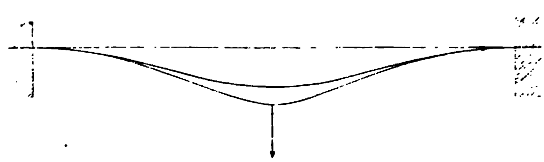


Fig. I. (*b*).

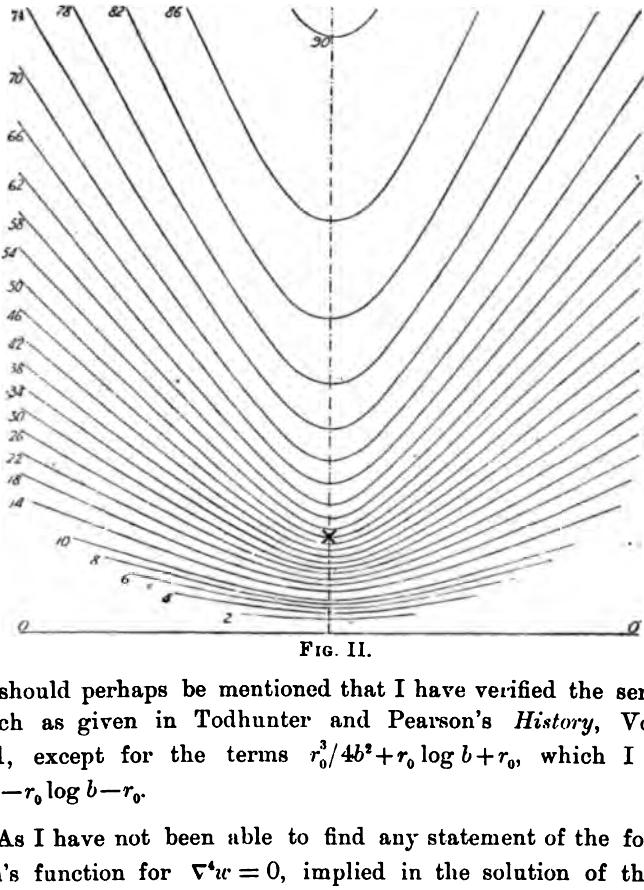


FIG. II.

It should perhaps be mentioned that I have verified the series of Clebsch as given in Todhunter and Pearson's *History*, Vol. II., § 1381, except for the terms $r_0^3/4b^2+r_0\log b+r_0$, which I make $r_0^3/4b^2-r_0\log b-r_0$.

5. As I have not been able to find any statement of the form of Green's function for $\nabla^4 w=0$, implied in the solution of the last section, it may be as well to put down the integrals which give the value of w within a circle, when the values of w and $\partial w/\partial n$ are given over the circumference.*

Taking $$w=\tfrac{1}{2}\left(\frac{h^2}{a^2}r'^2-r^2\right)-r^2\log\frac{hr'}{ar},$$

we readily deduce $$\nabla^2 w=2\frac{(a^2-h^2)^2}{a^2r^2},$$

$$\frac{\partial}{\partial n}\nabla^2 w=2\,(a^2-h^2)^2\frac{r^2+a^2-h^2}{a^3r^4},$$

as holding over the circumference.

* The method of Rayleigh, *loc. cit.*, solves the problem without the use of the Green's function.

Hence at the point O

$$w = -\frac{(a^2-h^2)^2}{4\pi a^2}\int \frac{\partial w}{\partial n}\,\frac{1}{r^2}\,ds + \frac{(a^2-h^2)^2}{4\pi a^2}\int w\,\frac{r^2+a^2-h^2}{r^4}\,ds$$

$$= -\frac{(a^2-h^2)^2}{4\pi a^2}\int \frac{\partial w}{\partial n}\,\frac{1}{r^2}\,ds + \frac{(a^2-h^2)^2}{2\pi a^2}\int w\,\frac{\cos\phi}{r^3}\,ds,$$

where ϕ is the inclination of δr to the outward normal δn.

It may be remarked that the Green's function for the equation $\nabla^4 w = 0$ within a sphere is

$$w = r - \frac{h}{a}\,r' + \frac{a^2-h^2}{2ah}\,\frac{a^2-\rho^2}{r'},$$

where ρ is the distance from the centre of the sphere, and the rest of the notation is the same as before.

There is a correspondingly simple formula for the stream function ψ for the slow motion of viscous liquid due to a source at the point O within a circular boundary together with an equal sink at the centre C. It is

$$\psi = \theta + \theta' - \vartheta + (a^2-\rho^2)\,\frac{\sin\theta'}{hr'},$$

where θ, θ', ϑ are the polar angles at O, O', C respectively. The stream function for the slow motion due to a line vortex at O in a circular cylinder is, with the same notation,

$$\psi = \log\frac{hr'}{ar} - (a^2-\rho^2)\,\frac{1}{h}\left(\frac{\cos\theta'}{r'} - \frac{1}{2}\,\frac{h}{a^2}\right).$$

Thursday, January 9th, 1902.

Dr. HOBSON, F.R.S., President, in the Chair.

Fourteen members and a visitor present.

The Rev. J. Cullen was admitted into the Society.

Major MacMahon, Vice-President, having taken the Chair, the President communicated a paper "On Non-uniform Convergence,

and the Integration of Series." Remarks upon the subject were made by Messrs. Larmor, Love, S. Roberts, Whittaker, and the Chairman.

The following papers were taken as read:—

On the Integrals of the Differential Equation

$$\frac{du}{\sqrt{f(u)}} + \frac{dv}{\sqrt{f(v)}} = 0,$$

where $f(x) \equiv ax^4 + 4bx^3 + 6cx^2 + 4dx + e,$

considered geometrically: Prof. W. S. Burnside.

On the Fundamental Theorem of Differential Equations: Mr. W. H. Young.

The following presents were made to the Library:—

"Educational Times," January, 1902.

"Indian Engineering," Vol. xxx., Nos. 21–24, Nov. 23–Dec. 14, 1901.

Dickson, L. E.—"Theory of Linear Groups in an Arbitrary Field," 4to; 1901.

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. lix., No. 453; 1901.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxv., Heft 12; Leipzig, 1901.

"Rendiconti del Circolo Matematico di Palermo," Tomo xv., Fasc. 5, 6; 1901.

"Bulletin of the American Mathematical Society," Series 2, Vol. viii., No. 3; New York, 1901.

"Bulletin des Sciences Mathématiques," Tome xxv., Nov., 1901; Paris.

"Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche," Vol. vii., Fasc. 8–11, Aug.–Nov., 1901; Napoli.

"Archives Néerlandaises," Série 2, Tome iv., Liv. 4, 5; La Haye, 1901.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. x., Fasc. 11; Roma, 1901.

"Proceedings of the Physical Society," Vol. xvii., Pt. 7; Dec., 1901.

"Tōkyō Sūgaku-Butsurigaku Kwai Kiji," Maki 8, Dai 6; 1901.

"Periodico di Matematica," Anno xvi., Fasc. 3; Livorno, 1901.

"Supplemento al Periodico di Matematica," Anno v., Fasc. 2; Livorno, 1901.

On the Integrals of the Differential Equation

$$\frac{du}{\sqrt{f(u)}}+\frac{dv}{\sqrt{f(v)}}=0,$$

where $f(x) \equiv ax^4+4bx^3+6cx^2+4dx+e,$

considered Geometrically. By W. Snow Burnside. Received January 1st, 1902. Communicated January 9th, 1902.

If the coordinates of a point on a conic be expressed as quadric functions of a single variable ϕ, viz.,

$$\left.\begin{aligned} kx &= a_0\phi^2+2b_0\phi+c_0 \\ ky &= a_1\phi^2+2b_1\phi+c_1 \\ kz &= a_2\phi^2+2b_2\phi+c_2 \end{aligned}\right\}, \tag{1}$$

the equation of the chord joining two points u, v on the conic is

$$\begin{aligned} &x\left\{(a_1b_2)\,uv+(a_1c_2)\frac{u+v}{2}+(b_1c_2)\right\} \\ &+y\left\{(a_2b_0)\,uv+(a_2c_0)\frac{u+v}{2}+(b_2c_0)\right\} \\ &+z\left\{(a_0b_1)\,uv+(a_0c_1)\frac{u+v}{2}+(b_0c_1)\right\}=0, \end{aligned}$$

and, if this chord touch any conic, we obtain an equation $\Sigma=0$ quadric in the variables uv and $u+v$, whence we may write Σ in the following forms:—

$$\Sigma = L_0v^2+2L_1v+L_2 = M_0u^2+2M_1u+M_2,$$

where L_0, L_1, L_2 are quadric functions of u, and M_0, M_1, M_2 are quadric functions of v.

Now, differentiating the equation $\Sigma=0$,

$$\frac{d\Sigma}{du}\,du+\frac{d\Sigma}{dv}\,dv=0,$$

which, in virtue of the equation $\Sigma=0$, may be written in the form

$$\sqrt{M_1^2-M_0M_2}\,du+\sqrt{L_1^2-L_0L_2}\,dv=0.$$

Again, since Σ is symmetrical in u and v, we obtain an equation of the form

$$\frac{du}{\sqrt{F(u)}}+\frac{dv}{\sqrt{F(v)}}=0.$$

We have thus arrived at a differential equation of the same type as

$$\frac{du}{\sqrt{f(u)}}+\frac{dv}{\sqrt{f(v)}}=0,$$

the integral of the former being $\Sigma=0$. We now proceed to show that, if the two conics are properly selected, $F(u)$ becomes identical with $f(u)$, no reductions being required.

The conics which serve our purpose are

$$U\equiv ax^2+cy^2+ez^2+2dyz+2czx+2bxy,$$
$$V\equiv y^2-4zx,$$

and we proceed to find the condition that the chord u, v of V should touch $U-\rho V$.

Let the tangential equation of $U-\rho V$ be written in the form

$$\Sigma=\Sigma_0-2\rho\Phi+\rho^2\Sigma_1=0;$$

then $\Sigma_0=(ce-d^2)\lambda^2+(ae-c^2)\mu^2+(ac-b^2)\nu^2$

$$+2(bc-ad)\mu\nu+2(bd-c^2)\nu\lambda+2(cd-be)\lambda\mu,$$

$$2\Phi=e\lambda^2+4c\mu^2+a\nu^2-4b\mu\nu+2c\nu\lambda-4d\lambda\mu,$$

$$\Sigma_1=4(\gamma\alpha-\beta^2),$$

where, if $\lambda x+\mu y+\nu z$ be the chord joining the points u and v on V, $\lambda=1$, $\mu=-\dfrac{u+v}{2}$, $\nu=uv$, the coordinates of a point on V, being given by the equations $\dfrac{x}{\phi^2}=\dfrac{y}{2\phi}=z$,

the simplest form of equations (1).

Differentiating Σ and supposing ρ variable, we have now

$$\tfrac{1}{2}d\Sigma=\sqrt{M_1^2-M_0M_2}\,du+\sqrt{L_1^2-L_0L_2}\,dv+\sqrt{\Phi^2-\Sigma_0\Sigma_1}\,d\rho,$$

since Σ is a quadric function of each of the variables u, v, ρ considered separately.

We proceed to reduce the last equation to the form

$$d\Sigma=i\sqrt{\Delta(\rho)f(v)}\,du+i\sqrt{\Delta(\rho)f(u)}\,dv+\sqrt{f(u)f(v)}\,d\rho,$$

where
$$i^2 = -1,$$
$$\Delta(\rho) = 4\rho^3 - I\rho + J,$$
the discriminant of the conic $U - \rho V$.

In order to calculate $L_1^2 - L_0 L_2$, let the tangential form of $U - \rho V$ be
$$\Sigma \equiv A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2C\nu\lambda + 2H\lambda\mu$$
$$= \left(Cv^2 - Fv + \frac{B}{4}\right)u^2 - \left\{Fv^2 - \left(2G + \frac{B}{2}\right)v + H\right\}u + B\frac{v^2}{4} - Hv - A,$$
since
$$\lambda = 1, \quad \mu = -\frac{u+v}{2}, \quad \nu = uv,$$
whence
$$4(L_0L_2 - L_1^2) = (BC - F^2)v^4 + 4(FG - CH)v^3$$
$$+ 2\{(HF - BG) + 2(AC - G^2)\}v^2$$
$$+ 4(HG - AF)v + AB - H^2,$$
every term of which is, from the theory of determinants, divisible by
$$\Delta(\rho) \equiv 4\rho^3 - I\rho + J,$$
the discriminant of $U - \rho V$; whence, reducing, since
$$HF - BG = \Delta(\rho)(c + 2\rho),$$
$$AC - G^2 = \Delta(\rho)(c - \rho),$$
$$4(L_0L_2 - L_1^2) = \Delta(\rho)(av^4 + 4bv^3 + 6cv^2 + 4dv + e) = \Delta(\rho) f(v).$$
Similarly,
$$4(M_0M_2 - M_1^2) = \Delta(\rho) f(u).$$
It remains now only to reduce the coefficient of $d\rho$ in $d\Sigma$. And, since, from the theory of a system of two conics, it is, when equated to zero, the tangential equation of the points of intersection of the conics U and V, viz., the points a, β, γ, δ, when
$$f(x) = (x - a)(x - \beta)(x - \gamma)(x - \delta);$$
so we have
$$4(\Phi^2 - \Sigma_0\Sigma_1) \equiv \kappa \prod_1^4 (\lambda x_r + \mu y_r + \nu z_r).$$
By comparing the coefficients of ν^4 on both sides of this equation,
$$a^2 = \kappa . z_1 z_2 z_3 z_4.$$
Again,
$$\lambda \frac{x_r}{z_r} + \mu \frac{y_r}{z_r} + \nu = 1 . a^2 - 2\frac{u+v}{2} a + uv = (a - u)(a - v);$$
therefore, finally,
$$4(\Phi_2 - \Sigma_0\Sigma_1) = f(u) f(v).$$

Now, substituting these values for the coefficients of du, dv, $d\rho$ in $d\Sigma$ and changing the sign of $\rho = -\rho'$, we have

$$\Sigma \equiv \Sigma_0 + 2\rho'\Phi + \rho'^2\Sigma_1 = 0$$

which is a particular integral of the differential equation

$$\frac{du}{\sqrt{f(u)}} + \frac{dv}{\sqrt{f(v)}} - \frac{d\rho'}{\sqrt{4\rho'^3 - I\rho' - J}} = 0 \qquad \text{(A)}$$

and the general integral of

$$\frac{du}{\sqrt{f(u)}} + \frac{dv}{\sqrt{f(v)}} = 0,$$

when ρ' is an arbitrary constant.

When $\Delta(\rho_1) = 0$, the above investigation requires to be modified; for in this case

$$\Sigma \equiv \Sigma_0 - 2\rho_1\Phi + \rho_1^2\Sigma_1$$

is a perfect square of the form

$$\kappa\,(\lambda x_1 + \mu y_1 + \nu z_1)^2,$$

where x_1, y_1, z_1 are the coordinates of one of the vertices of the common self-conjugate triangle of the conics U and V, and, if this point be the intersection of the common chords (β, γ) and (α, δ),

$$\frac{x_1}{\beta\gamma\,(\alpha+\delta) - \alpha\delta\,(\beta+\gamma)} = \frac{y_1}{2\,(\beta\gamma - \alpha\delta)} = \frac{z_1}{\beta+\gamma-\alpha-\delta}.$$

$$\lambda x_1 + \mu y_1 + \nu z_1 = 0$$

then becomes

$$(\beta+\gamma-\alpha-\delta)\,uv - (\beta\gamma-\alpha\delta)(u+v) + \beta\gamma\,(\alpha+\delta) - \alpha\delta\,(\beta+\gamma) = 0.$$

In this way, corresponding to the three roots of $\Delta(\rho) = 0$, we obtain three particular integrals of the equation

$$\frac{du}{\sqrt{f(u)}} + \frac{dv}{\sqrt{f(v)}} = 0,$$

which can be easily verified directly, and when $u = v$ these integrals become the three quadratic factors of the sextic covariant of $f(u)$.

When the chord (uv) becomes a tangent to V it is interesting to ascertain how the previous results are modified. Therefore making $u = v$ in the equation

$$\Sigma \equiv \Sigma_0 + 2\rho'\Phi + \rho'^2\Sigma_1 = 0,$$

Σ_0 becomes the Hessian of $f(u)$, 2Φ becomes $f(u)$, and Σ_1 vanishes; whence

$$\rho' = -\frac{H(u)}{f(u)}.$$

Again, the differential equation (A) becomes

$$2\frac{du}{\sqrt{f(u)}} = \frac{d\rho'}{\sqrt{4\rho'^3 - I\rho' - J}},$$

which is the general elliptic differential reduced to Weierstrass's canonical form by Hermite's substitution.

A very complete discussion of this subject will be found in chap. v. of Greenhill's *Elliptic Functions*, from the analytic side, and I offer this geometrical view of the matter only in order to show how it depends on the contravariants of the conics U and V.

On the Fundamental Theorem of Differential Equations. By W. H. Young. Received and communicated January 9th, 1902.

The fundamental theorem of the modern theory of differential equations is Cauchy's existence theorem, dealing with the existence and uniqueness of a set of integrals satisfying given initial conditions, and the holomorphic character of the solution. This theorem has been stated in very precise language, and proved in various ways, by Picard and Painlevé, but some doubt has been expressed as to whether their proofs are rigorous. It has been suggested, in fact, that it has not been conclusively demonstrated that the holomorphic solution is unique even in the simplest case which can arise.

In the following note* it is proposed to give a brief account of the theorem in question, and to examine an example which has been put forward as typical of a large class of cases where the theorem fails.

* The note is substantially what I wrote in January, 1899, but did not publish, as I expected Painlevé or Picard to take the matter up. The former has now done so, but his treatment is too general to appeal to English readers. Indeed he does little more than repeat at length his previous definitions.

Taking the case of a single differential equation of the first order and first degree, Cauchy's theorem may be thus stated:—

Given a differential equation

$$\frac{dy}{dx}=f(x,\,y), \tag{1}$$

and a pair of values a, b for which the function f is holomorphic, there exists one, and only one, integral of the equation which approaches the value b when x approaches the value a, and this integral is holomorphic.*

We add as gloss: When we say that "y approaches the value b when x approaches the value a," we mean that, if we consider small circles of radii ϵ and σ round the points b of the y plane and a of the x plane, and make x move up towards a along any path which, from and after a certain fixed point, enters and remains in the circle of radius σ, then y moves along a certain curve which from and after a certain fixed point (to be determined) enters and remains in the circle of radius ϵ; and this is to be true however small σ and ϵ are taken, provided only they are fixed. Such curves may be called "curves of approach" to the point in question. The student of precise theory of functions will recognise that this is, in fact, only a gloss and not a hypothesis, since in treating of the value of a function at a point, or the behaviour of a function in the neighbourhood of a point, we are working in the small (*im Kleinen*), and the geometry we can apply is only differential geometry. If we did not work in the small, we should find ourselves constantly hampered by quite unnecessary complications. For instance, starting from a point on a Riemann's surface, and considering such a commonplace function as an Abelian function with three or more periods having a given value at that point, the value at a neighbouring point is determinate, because we are working "in the small." But, if we allow the moving point to wander about at will on the Riemann's surface between the initial and final point, the Abelian function can, by a proper choice of path, be made to have a value at the final point as near as we please to any given value whatever. Thus the student who refused to work "in the small" would be tempted to say that such a function had an essential singularity or "point of indeterminateness" at every point of the Riemann's surface! The uselessness of such a mode of expression is self-evident.

* By "holomorphic at a point c" we mean that the function is developable in a series of positive integral powers of $(x-c)$ in the neighbourhood of that point.

Notice that there are virtually three independent statements in the theorem :—

(i.) That a solution exists, satisfying the given conditions, which is holomorphic at the point $x = a$.

(ii.) That this solution is the only one which is holomorphic at the point.

(iii.) That there is no non-holomorphic solution satisfying the conditions.

The first two statements are due to Cauchy ; proofs of the third have been given by Picard and Painlevé.

We confine our attention to (iii.), referring to any of the many treatises on analysis for an account of (i.) and (ii.).

Picard's elegant proof is given in his *Traité d'Analyse*, and is of this nature: He assumes the truth of (i.) and (ii.) for partial differential equations and deduces the truth of (iii.) for our case.

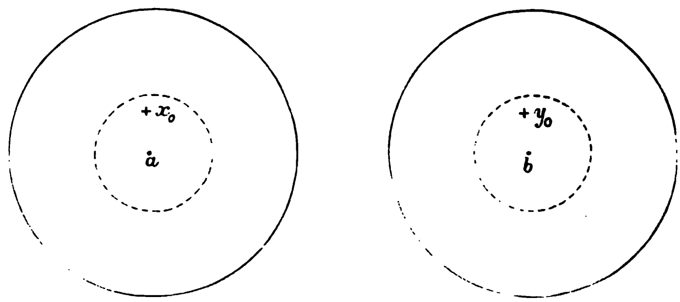
Painlevé has given more than one proof. His first proof is given on pp. 19, 20 of his Stockholm lectures. It is somewhat concisely stated, and its force may therefore be easily missed. It requires a knowledge of the domain in which the Cauchy integral is proved to be holomorphic.

Suppose $f(x, y)$ holomorphic for

$$|x-a| \leqslant r \quad \text{and} \quad |y-b| \leqslant \rho, \tag{2}$$

and let M be the modulus of its upper limit in this region ; then it is known that Cauchy's integral is holomorphic within a circle centre a and radius λ, where

$$\lambda = r\left(1-e^{-\rho/3Mr}\right). \tag{2'}$$



It at once follows that this is a Cauchy integral which has the value y_0 when $x = x_0$, and y_0 being *any* points within the circles (dotted

in figure) centres a and b and radii $\frac{1}{2}r$ and $\frac{1}{2}\rho$, and is holomorphic within a circle of radius $\frac{1}{2}\lambda$ and centre x_0. Call these dotted circles C_1 and Γ_1.

We can now give the substance of Painlevé's proof.

Let L be a path of approach* to the point $x = a$.

Is it possible that for every such path, or for any one such path, a solution $y = \phi(x)$ exists satisfying the given couditions, and non-holomorphic at the point $x = a$?

In virtue of the hypothesis with regard to L on p. 235, we can evidently find a point H of L so near a that—

(1) Its distance from a is $< \frac{1}{2}\lambda$, and *a fortiori* $< \frac{1}{2}r$.

(2) The corresponding point y is within Γ_1.

(3) These statements are true for all points on L between a and this point H.

Assume:

(4) That among all these points one at least exists for which $\phi(x)$ is holomorphic.

(*I.e.*, assume that the portion of L between H and a is not singular for ϕ.)

Call this point x_0 and denote the corresponding y point by y_0. Then ϕ is the Cauchy integral corresponding to these values; for it is holomorphic at $x = x_0$.

Now, since $f(x, y)$ is holomorphic for

$$|x-x_0| < \tfrac{1}{2}r, \quad |y-y_0| < \tfrac{1}{2}\rho,$$

it follows that the Cauchy integral corresponding to the pair of values (x_0, y_0) is holomorphic at all points of the circle whose radius is $\frac{1}{2}\lambda$ and centre x_0. But, by (1), p. 237, this circle includes the point a. Hence ϕ is holomorphic at the point a. Hence it is the Cauchy integral at the point a, and the hypothesis that it could be non-holomorphic there will not hold.

The only doubt then that can exist is:—

Is there perhaps a function of x which satisfies the differential equation and which is non-holomorphic at all points of an arc of a curve of approach to a including its extremity (or asymptotic point) a, and which tends towards b as x moves on the curve towards a?

If this be the case, then a solution exists which has the value b for

* The path L may have the point a as asymptotic point, and its length may increase indefinitely as x tends to a.

one, at any rate, of its values at a, and which is not the Cauchy integral.

Are there any functions of a single complex variable which are non-developable at all points of a certain curve lying in the region of existence of the function, and which none the less have a differential coefficient at each of these points?

We defer for a subsequent paper the discussion of this question. It will be seen shortly that it is not necessary to answer it for the purpose in hand if we make use of Painlevé's second method of treatment of the problem.

Let (x_0, y_0) be any pair of values in the domain in which $f(x, y)$ is holomorphic, and let the corresponding Cauchy integral be

$$y = \phi(x, y_0, x_0). \tag{A}$$

Let $(\bar{x}, \bar{y})$ be any pair of values of (x, y) satisfying equation (A), and therefore lying in the region of existence of ϕ, and in the domain in which f is holomorphic. Then, by the uniqueness of the Cauchy holomorphic solution, we get the same Cauchy integral (A) if we start with $(\bar{x}, \bar{y})$ instead of with (x_0, y_0), *i.e.*,

$$y = \phi(x, \bar{y}, \bar{x}),$$

is identical with (A). But (A) is satisfied by $x = x_0$, $y = y_0$; therefore

$$y_0 = \phi(x_0, \bar{y}, \bar{x}),$$

or, since $\bar{y}, \bar{x}$ were any pair, $y_0 = \phi(x_0, y, x)$ (A′)

is another way of writing (A).

Now give x_0 a definite numerical value a, and write

$$u = \phi(a, y, x), \tag{B}$$

where y of course no longer satisfies (A). Then u is a function of x and y which assumes the value b, when

$$x = a \quad \text{and} \quad y = b.$$

Change the dependent variable in our fundamental equation from y to u. We know that (B) is algebraically equivalent to

$$y = \phi(x, u, a). \tag{B′}$$

Hence our fundamental equation (1) becomes

$$\frac{\partial}{\partial x}\{\phi(x, u, a)\} + \frac{du}{dx}\frac{\partial}{\partial u}\{\phi(x, u, a)\} = f\{\phi(x, u, a), x\}.$$

But, since (A) satisfies (1), we have *identically*

$$\frac{\partial}{\partial x}\phi(x, y_0, x_0) = f\{\phi(x, y_0, x_0)\, x\}$$

for all values of x, y_0, and x_0; and therefore

$$\frac{\partial}{\partial x}\phi(x, u, a) \equiv f\{\phi(x, u, a), x\}.$$

Hence $$\frac{du}{dx}\frac{\partial}{\partial u}\phi(x, u, a) = 0;$$

and therefore, since ϕ certainly contains u, we have

$$\frac{du}{dx} = 0 \qquad (1')$$

as the new form of our equation (1).

Moreover, since when $x = a$, $y = b$ we have $u = b$, it follows that our initial conditions are

$$u = b, \quad \text{when} \quad x = a, \qquad (2')$$

or, more strictly, that, as x approaches a, u has to approach b in the usual way.

The obvious solution of equation (1′) subject to condition (2′) is

$$u = b;$$

whence, by (B′), $$y = \phi(x, b, a);$$

in other words, we are led by virtue of the uniqueness of the holomorphic solution back to equation (A).

Has the equation (1′) another solution?

In other words—

Can a function S be found and an arc Q (having x_0 for its limiting point) such that at all points of a certain domain in which Q lies the function S has a differential coefficient which is zero, but which along the arc Q is not developable?

This is, of course, impossible; for it is known that no function exists which has its differential coefficient zero at all points of a domain, except a constant.

For any point P of the domain can be joined to a fixed point A of the domain by means of a continuous line, consisting of a finite number of straight lines lying entirely within the domain. Since along each of these straight lines the differential coefficient is zero, we know, by the *Mengenlehre*, that the function is constant, and has therefore the same value at P as at A.

Thus the theorem (iii.) is completely proved.

We now proceed to discuss in some detail the typical example to which we referred in our opening remarks.

Consider the differential equation*

$$\frac{dy}{dz} = \frac{y^2}{y^2+z-y}, \tag{1}$$

and let us seek for an integral, other than the trivial one $y=0$, which satisfies the condition of approaching the value 0 as z approaches the value c. According to Cauchy's theorem no such integral exists. It has been contended,† however, that a non-holomorphic integral exists which may be constructed as follows.

Take the complete integral of (1),

$$z = y + ae^{-1/y}, \tag{2}$$

where a is a constant of integration, and, writing

$$-\frac{1}{y} = -L + 2x\pi i, \tag{3}$$

where L is any particular chosen logarithm of $\frac{a}{c}$, let us give to x a series of integral values each numerically greater than the preceding; ultimately y and $z-c$ may in this way both be made as small as we please, and therefore it is asserted the non-holomorphic integral (2) satisfies the condition required (and this for all values of the arbitrary constant).

To examine the question completely, let all the quantities involved be considered as complex. We have three planes: an x-plane, a y-plane, and a z-plane. The x-plane is connected with each of the other planes by an analytical transformation. *The transformation of the x-plane into the y-plane is a simple inversion, given by the equation* (3), *the centre of inversion in the y-plane being the origin, and in*

* See Forsyth, *Theory of Differential Equations*, Part 2, Vol. II., p. 81, where the equation in question is discussed. The notation adopted is only slightly different. We have written z for $\frac{ab-a\beta}{b}z$, y for βy, a for $\frac{A}{b}$, c for $\frac{ab-a\beta}{\beta}c$, and, for obvious reasons, x for k. We have omitted Forsyth's first differential equation, which is connected with the other by the substitution

$$w = -\frac{a}{\beta}z + y.$$

A slight oversight with respect to the constant A is amended.

† Forsyth, *loc. cit.*

the x-plane the point $\dfrac{L}{2\pi i}$. Using polar coordinates referred to the centres of inversion r, θ in the x-plane, and ρ, ϕ in the y-plane, the equation (3) is equivalent to the two equations

$$\left.\begin{aligned} r\rho &= \frac{1}{2\pi} \\ \phi &= \frac{\pi}{2} - \theta \end{aligned}\right\}. \tag{3'}$$

Corresponding then to the *interior* of a small circle, centre the origin and radius η, in the y-plane, we have the *exterior* of a large circle, centre $\dfrac{L}{2\pi i}$ and radius $\dfrac{1}{2\pi\eta}$, in the x-plane.

Corresponding to that part of any curve of approach to the origin in the y-plane which lies inside the small circle in the y-plane, we have a portion of a curve exterior to the large circle in the x-plane and going off to infinity.

Again, from equations (2) and (3), we have

$$z = y + ce^{2x\pi i}, \tag{4}$$

where

$$y = \frac{1}{L - 2x\pi i}. \tag{3}$$

These equations define a transformation of the x-plane into the z-plane, and of the z-plane into the x-plane.

We have then to consider whether it is possible to find corresponding sets of values of z and y, the one set forming a curve of approach to the point c in the z-plane, and the other a curve of approach to the point O in the y-plane. If so, then, by definition, quantities ϵ and η can be found, such that, if we draw a circle centre c and radius ϵ in the z-plane, and a circle with centre O and radius η in the y-plane, both curves, on entering these circles, approach and never recede from the respective centres.

Assume the possibility of the point at issue, and inquire as to the region or regions of the x-plane to which the x-point is confined when the z-point has entered its circle of radius ϵ. By hypothesis the y-point must then have entered the corresponding circle radius η.

Let us introduce an auxiliary point z' given by

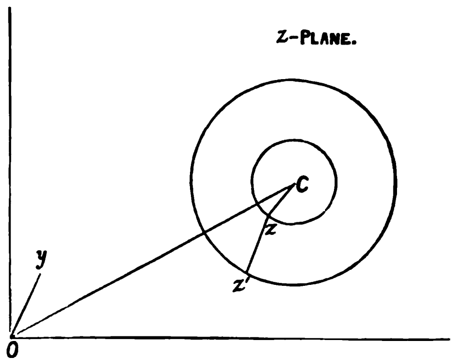
$$z' = ce^{2x\pi i}, \tag{5}$$

so that

$$z = y + z'. \tag{5'}$$

This shows us that (the y- and z-planes being taken for the moment

as coinciding) the straight line $z'z$ is equal and parallel to the radius vector of y, and hence that z' must lie within a circle centre c and radius $\epsilon+\eta$.



Writing then $$x = x_1 + ix_2,$$

and considering the length of the radius vector of z', we have

$$|c| - \epsilon - \eta \leqslant \text{mod of } z' \leqslant |c| + \epsilon + \eta,$$

or $$|c| - \epsilon - \eta \leqslant |c|\, e^{-2x_2\pi} \leqslant |c| + \epsilon + \eta.$$

This shows that

I. *x_2 must be a small quantity lying between the limits $-\lambda_1$ and λ_2, where λ_1 and λ_2 are determinate small real positive quantities, viz.,*

$$-2\pi\lambda_1 = \textit{real logarithm of } 1 - \frac{\epsilon+\eta}{|c|},$$

$$2\pi\lambda_2 = \textit{real logarithm of } 1 + \frac{\epsilon+\eta}{|c|},$$

and evidently tend towards zero when either or both of the small quantities ϵ and η tend towards zero.

Considering, on the other hand, the vectorial angle of z', we see that (since z' must lie between the tangents from the origin to the circle of radius $\epsilon+\eta$):

II. *$2x_1\pi$ must differ from an integral multiple of 2π by a quantity less in absolute magnitude than the small angle—call it $2\pi\sigma$—whose sine is $\dfrac{\epsilon+\eta}{|c|}$.*

From I. and II. it follows that the point x must always, when

the z-point has entered the circle of radins ϵ, and the y-point the corresponding circle of radius η, lie in one or other of certain small parallelograms of the x-plane, the breadth of each parallelogram being 2σ, and the height $(\lambda_1+\lambda_2)$, and each parallelogram containing one (and of course only one) integral point of the real axis.

Further, as the x-point, and therefore also the y-point, moves along its curve of approach, both points enter circles of smaller and smaller radii. In other words, in the above investigation, we may diminish ϵ and η. It follows therefore, in accordance with I. and II., that the x-point remains in its parallelogram and moves towards the integral point belonging to it.

But, as remarked on p. 241 in connexion with equation (3′), the x-point must move in such a way as to remain outside the large circle in the x-plane whose centre is $\dfrac{L}{2\pi i}$, and whose radius is $\dfrac{1}{2\pi\eta}$, and therefore constantly increases as η decreases. The figure shows that this is inconsistent with the point remaining inside its parallelogram.

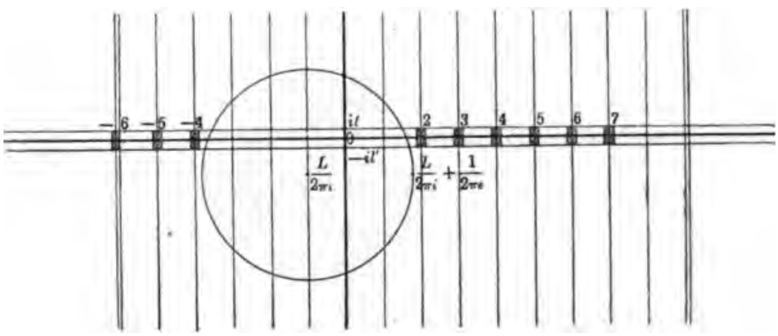


FIG. 1.

It has thus been conclusively shown by a *reductio ad absurdum* that, as the z-point moves along a curve of approach to c, the y-point cannot move along a corresponding curve of approach to its origin.

If we could make the z-point move *discontinuously*, we could make the x-point jump from parallelogram to parallelogram. The points of the parallelograms suitable for this purpose are of course the integral points of the real axis; for all the other points come to lie outside the parallelograms as ϵ and η diminish. If we make x jump along these integral points, z and y each jump along the points of an *abzählbare Menge* having the desired goals as points of condensation.

But our investigation shows that, if we attempt to draw any curve of approach through one of these *Mengen*, the corresponding curve will pass through the other *Menge*, but between each pair of points it will recede to a finite distance from the goal and come back again; so that it is not a "curve of approach."

It is of interest to take a particular curve of approach in the one plane, and see what happens to the curve corresponding to it in the other plane, and why it is not a curve of approach. The simplest case is obtained if we consider a curve of approach in the y-plane and inquire how the z-point must move.

Let us move along the real axis in the x-plane, and therefore in the y-plane move along the circle (Fig. 2)

$$y_1^2+y_2^2-\frac{y_1}{L_1}=0,$$

where
$$y=y_1+iy_2,$$

and L_1 is the real part of L.

The point z will simultaneously describe the spiral

$$z=ce^{2x\pi i}+\frac{1}{L-2x\pi i},$$

where x is now a real parameter.

This spiral lies entirely outside or inside the circle

$$z=ce^{2x\pi i} \quad \text{or} \quad z_1^2+z_2^2=|c|^2,$$

according as $\dfrac{1}{L}$ is positive or negative.

Fig. 3 shows the spiral for x positive. For x negative we have to reflect the curve in the real axis. The dotted circle is the

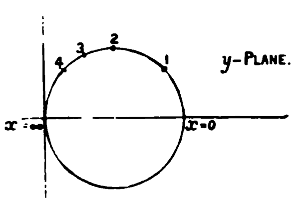


FIG. 2.

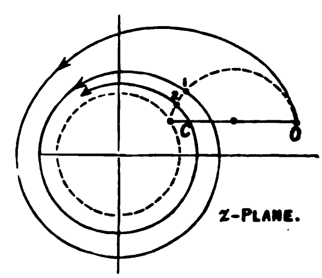


FIG. 3.

circle to which the whorls of the spiral approximate. The dotted

semicircle cuts out the positive integral points. The negative integral points would be cut out by the lower semicircle of the same circle whose equation is

$$(z-c_1)^2+(z_2-c_2)^2 = \frac{z_1-c_1}{L_1}.$$

Non-uniform Convergence, and the Integration of Series. By E. W. Hobson, Sc.D., F.R.S. Read and received January 9th, 1902.

If the terms of an infinite series are functions of a real variable which are all continuous in a given interval taken as the field of the variable, and the series converges at every point of the interval, it is well known that the convergence of the series must be non-uniform in the neighbourhood of any point at which the sum of the series is discontinuous, but that non-uniformity of convergence in the neighbourhood of a point does not necessarily imply discontinuity of the sum at that point. In the case in which the sum of the series is continuous throughout the whole interval, the most general possible distribution of points of non-uniform convergence of the series has been obtained by Osgood.* He has shown that the points at which the measure of non-uniform convergence† exceeds an arbitrarily fixed positive number form a closed aggregate, non-dense in the given interval, and that the points at which the convergence is uniform form an everywhere dense aggregate.

In a very remarkable memoir,‡ Baire has proved that the sum of a series such as has been described is at most a point-wise discontinuous function, *i.e.*, in any sub-interval points can be found at which the function is continuous. The distribution of points of non-uniform convergence, which is of fundamental importance in the question of the integration of the series, was, however, not considered by Baire. In the present paper, it is shown by a method on the lines of that of Baire, that the most general dis-

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* See his paper, "Non-uniform Convergence and the Integration of Series," *Amer. Jour. of Math.*, Vol. XIX., 1897.

† This term will be explained in the course of the paper.

‡ "Sur les fonctions de variables réelles," *Annali di Math.*, Vol. III., 1899.

tribution of points of non-uniform convergence is the same as in the special case considered by Osgood, in which the sum of the series is everywhere continuous in the interval of convergence. For the sake of completeness, a simplified form of proof of Baire's result has been incorporated in the paper.

Osgood has proved (*loc. cit.*) that, in case the sum is everywhere continuous, it is a sufficient condition that the integral of the sum may be represented by the sum of the integrals of the terms of the series, that there should be no points in the interval of integration at which the measure of non-uniform convergence is indefinitely great. When the sum of the series is a point-wise discontinuous function, it is not certain that the sum is integrable; it was first pointed out by H. J. S. Smith* that a point-wise discontinuous function is not necessarily integrable. It is here shown that, provided the sum of the series is integrable, the integral is the sum of the integrals of the terms of the series, if a condition corresponding to that of Osgood, is satisfied, viz., that the measure of non-uniform convergence is everywhere less than some fixed finite number. As a matter of convenience, the expression "non-uniformly convergent at a point" is throughout used instead of the more accurate expression "non-uniformly convergent in the neighbourhood of a point."

1. Suppose $$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots$$

is a series which converges for all values of the real variable x, which lie in the interval (a, b); each of the functions $u_n(x)$ is supposed to be continuous at every point of the given interval.

The sum-function $s_n(x)$ is defined by

$$s_n(x) = u_1(x)+u_2(x)+\ldots+u_n(x),$$

and $L_{n=\infty} s_n(x)$ is denoted by $s(x)$; the difference $s(x)-s_n(x)$ may be denoted by $R_n(x)$, which is called the remainder function.

The functions $s_n(x)$, $R_n(x)$ may be regarded as functions of the two variables x, n, and are at present only defined for positive integral values of n; if we take y instead of n as variable, where

$$n = \frac{1}{y},$$

the functions $s_n(x)$, $R_n(x)$ may be written as $s(x, y)$, $R(x, y)$. In

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* *Proc. Lond. Math. Soc.*, Vol. VI.

order to define $s(x, y)$, $R(x, y)$, for values of y which do not correspond to positive integral values of n, we may suppose that for a value y which lies between the values y_m and y_{m+1}, which correspond to the consecutive integral values m, $m+1$, of n, the functions are defined by

$$s(x, y) = \frac{y-y_m}{y_{m+1}-y_m}\, s(x, y_{m+1}) + \frac{y_{m+1}-y}{y_{m+1}-y_m}\, s(x, y_m),$$

$$R(x, y) = \frac{y-y_m}{y_{m+1}-y_m}\, R(x, y_{m+1}) + \frac{y_{m+1}-y}{y_{m+1}-y_m}\, R(x, y_m);$$

thus the functions are linear functions of y between the values y_m and y_{m+1}.

Further, we assume that

$$R(x, 0) = 0, \quad s(x, 0) = s(x);$$

the functions $s(x, y)$, $R(x, y)$ are now defined for all points in the rectangle bounded by the four straight lines $x = a$, $x = b$, $y = 0$, $y = 1$.

The functions $s(x, y)$, $R(x, y)$ thus defined may be called the transformed sum-function and the transformed remainder-function, respectively, for the given series. Both functions are continuous with respect to y at every point in the rectangle; that this is the case for points on the boundary $y = 0$ follows from the condition of convergency of the series, that for a constant x the limits of $s_n(x)$ and $R_n(x)$, as n is indefinitely increased, are $s(x)$ and zero respectively, and thus that

$$\mathrm{L}_{y=0}\, s(x, y) = s(x) = s(x, 0)$$

and $$\mathrm{L}_{y=0}\, R(x, y) = 0 = R(x, 0).$$

The function $s(x, y)$ is also continuous with regard to x for all values of y in the rectangle, except on the boundary $y = 0$; its nature on this boundary is here to be investigated. It will be observed that the function $R(x, y)$ does not fall under the general class of functions considered by Baire, which are everywhere continuous with regard to y, and with respect to x along a series of lines parallel to the x-axis cutting the y-axis in an everywhere dense aggregate of points; it may be discontinuous with respect to x at points lying on all lines parallel to the x-axis.

2. Through a point $P(x, 0)$ on the boundary $y = 0$, draw a line parallel to the axis of y, and of any length ρ, (PP_1); the fluctuation

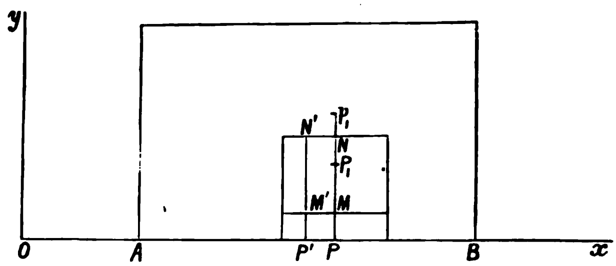

Fig. 1.

of the function $s(x, y)$ in this straight line has a value whose limit is zero when ρ is indefinitely diminished; let σ be any positive number; then there is an upper limit of those values of ρ which are such that the fluctuation of $s(x, y)$ in the line ρ is $\leqq \sigma$; we denote this upper limit by $\lambda_\sigma(x)$; thus, if $\rho \leqq \lambda_\sigma(x)$, the fluctuation in ρ is $\leqq \sigma$, and, if $\rho > \lambda_\sigma(x)$, the fluctuation is $> \sigma$.

It will now be shown that, corresponding to any arbitrarily small fixed number ϵ, an interval 2δ can be found on the x-axis, with P as its middle point, such that for all points P' within this interval the value of $\lambda_\sigma(P')$ is less* than $\lambda_\sigma(x) + \epsilon$.

Let $PP_1 = \lambda_\sigma(x)$, and let $P_1 p_1 = \epsilon$; then in Pp_1 the fluctuation of the function $s(x, y)$ is greater than σ, say $\sigma + \kappa$; if a number $\kappa_1 < \kappa$ be taken, two points M, N, neither of which coincides with P, can be found, such that $|s(M) - s(N)| > \sigma + \kappa_1$. Since $s(x, y)$ is continuous with regard to x, at both M and N, straight lines of equal lengths 2δ can be drawn through these points parallel to the x-axis, such that in each of them the fluctuation of $s(x, y)$ is less than $\frac{1}{2}\kappa_1$, and such that M, N are their middle points.

Let P' be any point on the x-axis whose distance from P is less than δ; draw $P'M'N'$ to meet the parallels through M and N in M', N'. We then have

$$|s(M) - s(N)| > \sigma + \kappa_1,$$
$$|s(N') - s(N)| < \tfrac{1}{2}\kappa_1,$$
$$|s(M') - s(M)| < \tfrac{1}{2}\kappa_1;$$

it follows that
$$|s(M') - s(N')| > \sigma.$$

* This property is named by Baire "semi-continuity of the function $\lambda_\sigma(x)$"; a semi-continuous function is not necessarily continuous.

Thus the value of $\lambda(P')$ must be less than the length $P'N'$, since the fluctuation of $s(x, y)$ in $P'N'$ is greater than σ; thus for any point P' in the interval 2δ with P as its middle point, we have

$$\lambda(P') < \lambda(P) + \epsilon.$$

Since $$R(x, y) = s(x) - s(x, y),$$

and $s(x)$ is constant with regard to y, it is clear that the function $\lambda_\sigma(x)$ for $R(x, y)$ is identical with the same function for $s(x, y)$.

3. In any sub-interval of the x-axis which contains the point $x(P)$, the essentially positive function $\lambda_\sigma(x)$ has a lower limit for all its values in the sub-interval; the limiting value of this lower limit as the sub-interval containing P is indefinitely diminished may be called the minimum of $\lambda_\sigma(x)$ at P; this minimum may be either

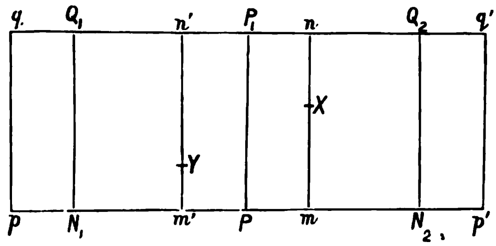

FIG. 2.

positive or zero—let us suppose that it has a positive value at P, and choose any positive number a less than this minimum. It is possible to find a neighbourhood pp' of P, such that for all points of pp' the value of $\lambda_\sigma(x)$ is greater than a; draw a rectangle on pp' as base and of height $a = PP_1$.

Since $s(x, y)$ is continuous at P_1, with respect to x, a neighbourhood Q_1Q_2 can be found for P_1 such that the fluctuation of $s(x, y)$ in Q_1Q_2 is less than an arbitrarily assigned number ϵ_1. Complete the rectangle $Q_1N_1N_2Q_2$. Let X, Y be any two points in the smaller of the two rectangles qp', Q_1N_2, and draw the straight lines mXn, $m'Xn'$; we have

$$|s(X) - s(Y)| \leqq |s(X) - s(n)| + |s(Y) - s(n')| + |s(n) - s(n')|$$
$$\leqq 2\sigma + \epsilon,$$

since the fluctuations in mn and $m'n'$ neither exceed σ.

It thus appears that on the supposition that the minimum of $\lambda_\sigma(x)$

at P is positive, an area Q_1N_2 or qp', whichever is the smaller, can be found such that the fluctuation of $s(x, y)$ considered as a function in the two-dimensional continuum (x, y) is in that area $\leqq 2\sigma+\epsilon$.

The above reasoning applies however small ϵ may be; thus we see that, if the saltus of $s(x, y)$ *at* the point P with reference to the continuum (x, y) is $>2\sigma$, then the minimum of $\lambda_\sigma(x)$ at P must be zero. By the saltus at P is meant the limit of the fluctuation in an area containing P when that area is indefinitely diminished.

To prove the corresponding theorem for $R(x, y)$, we have, since

$$R(m) = 0, \quad R(m') = 0,$$

$$|R(X)-R(Y)| \leqq |R(X)-R(m)| + |R(Y)-R(m')| \leqq 2\sigma$$

for all points X, Y in the rectangle $qpp'q'$; it follows that, if the saltus of $R(x, y)$ at P is greater than 2σ, the function $\lambda_\sigma(x)$ has its minimum at P equal to zero.

4. It will now be shown that in every sub-interval, however small, which forms part of the interval (a, b), there exist points at which the minimum of $\lambda_\sigma(x)$ is positive. Suppose, if possible, that at every point of the sub-interval (α_1, β_1) the minimum of $\lambda_\sigma(x)$ is zero; then, if ϵ_1 be any assigned number, a point P_1 can be found in (α_1, β_1) at which $\lambda_\sigma(P) < \frac{1}{2}\epsilon_1$; in virtue of what has been shown in § 3, a neighbourhood (α_2, β_2) of P_1 can be found such that at every point in it $\lambda_\sigma(x) < \lambda_\sigma(P_1)+\frac{1}{2}\epsilon_1 < \epsilon_1$.

Now consider a convergent sequence of decreasing numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$ which has its limit zero; by continuing the above reasoning, a series of sub-intervals $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n), \ldots$, each one enclosing the next, can be found, such that in any one sub-interval (α_n, β_n) at every point $\lambda_\sigma(x) < \epsilon_n$. It is well known that in such a series of sub-intervals there is at least one point P which is in the interior of all the sub-intervals; at this point P, $\lambda_\sigma(P) < \epsilon_n$, however great n may be; hence $\lambda_\sigma(P)$ is zero; but $\lambda_\sigma(x)$ is positive for every value of x; it therefore follows that it is impossible that at every point of a sub-interval (α_1, β_1) the minimum of $\lambda_\sigma(x)$ should be zero.

It has thus been shown that in every sub-interval belonging to (a, b) there exists one point, and therefore also an indefinitely great number of points, at which the minimum of $\lambda_\sigma(x)$ is positive.

The points of (a, b) at which the minimum of $\lambda_\sigma(x)$ is zero form a closed aggregate; for let P be a limiting point of this aggregate; then in any arbitrarily small sub-interval containing P there are an

indefinitely great number of points of the aggregate; hence in this sub-interval there are points at which $\lambda_\sigma(x)$ is as small as we please, and therefore the minimum of $\lambda_\sigma(x)$ at P is zero, or P is itself a point belonging to the aggregate; thus every limiting point of the aggregate of points at which the minimum of $\lambda_\sigma(x)$ is zero belongs to the aggregate, or the aggregate is a closed one. The aggregate is throughout the interval (a, b) non-dense, for, if it were dense in any sub-interval, since it is also closed, it would contain all the points of the sub-interval, and this has been shown not to be the case.

We see therefore that in any sub-interval (α, β) of (a, b), a sub-interval (α_1, β_1) can be found at every point of which the minimum of $\lambda_\sigma(x)$ is positive.

5. If we combine the results of the last two articles, we see that in every sub-interval of (a, b), a sub-interval (α_1, β_1) can be found such that at every point in it the fluctuations of the two functions $s(x, y)$, $R(x, y)$ with reference to the continuum (x, y) are less than 2σ.

If we take a convergent sequence of decreasing values of σ, $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ whose limit is zero, we see that in (α, β) a series of sub-intervals (α_1, β_1), (α_2, β_2), ..., (α_n, β_n), ... can be found, each enclosing the next one and such that at every point of (α_n, β_n) the fluctuations of the functions $s(x, y)$, $R(x, y)$ are each less than $2\sigma_n$; it follows that there is in (α, β) at least one point at which the fluctuations are both zero, or at which the functions are continuous.

We have thus obtained the result that in every sub-interval of (a, b) there exist points at which the functions $s(x, y)$, $R(x, y)$ are continuous with reference to the continuum (x, y).

We may state the result in the form that the points on the x-axis at which either of the two functions is continuous with reference to (x, y) form an everywhere dense aggregate.

It has been shown by Baire that the function $s(x, y)$ is such that in any area contained in the rectangle bounded by $x = a$, $x = b$, $y = 0$, $y = 1$, there are points at which the function is continuous with respect to (x, y), or, in other words, $s(x, y)$ is a point-wise discontinuous function; of this result we have, however, no need; for our purpose it is sufficient to consider the nature of the function along the x-axis. It will be observed that $R(x, y)$ may be discontinuous with respect to x at points lying on every line parallel to the x-axis, except on that axis itself.

6. At a point $(x, 0)$ of the interval (a, b), at which $s(x, y)$ is discontinuous with regard to (x, y), the function $s(x, 0)$ or $s(x)$ may or may not be discontinuous with respect to x; but, since the points of discontinuity of $s(x)$ must be included in those of $s(x, y)$, we have Baire's result that $s(x)$, the sum of the convergent series $\Sigma u(x)$, in which $u(x)$ is continuous in the interval (a, b), is at most a pointwise discontinuous function, that is to say, the points of continuity of $s(x)$ form an everywhere dense aggregate in (a, b).

We pass on to the consideration of the function $R(x, y)$; this function is zero for all values of x in the interval (a, b), when $y = 0$, but it has in general points of discontinuity with respect to (x, y) on the boundary $y = 0$.

Let P be a point in the interval (a, b) on the x-axis; draw a semicircle qpq' of radius ρ, with P as centre. The upper limit of $|R(x, y)|$ in this semicircle will have a value $\beta(\rho)$ which is a function of ρ, and which when ρ is indefinitely diminished has a limiting value β_P. If β_P is zero, then P is a point of continuity of $R(x, y)$, but, if β_P has any value different from zero, P is a point of discontinuity of $R(x, y)$.

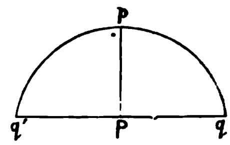

FIG. 3.

It is easily seen that a point P of discontinuity of $R(x, y)$ is a point in whose neighbourhood, or, as may be conveniently expressed, at which, the convergence of the series is non-uniform. At a point of uniform convergence of the series, in a sufficiently small neighbourhood NM, corresponding to an arbitrarily small prescribed positive number ϵ, a value m of n can be found such that, if $n \geq m$, then $|R_n(x)| < \epsilon$, for all values of x in MN; thus a rectangle $MNSR$ can be found such that for all points in it $|R(x, y)| < \epsilon$; within this rectangle semicircles with centre P can be described in which $|R(x, y)| < \epsilon$, and thus P would be a point of continuity of the transformed remainder function.

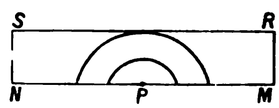

FIG. 4.

We have thus obtained the following theorem:—

If $u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$ be a series which converges for all values of x in an interval (a, b), and each term $u_n(x)$ is continuous throughout the interval, those points in whose neighbourhood the series converges uniformly form an aggregate which is everywhere dense throughout the interval (a, b).

7. The number β_x, which has been defined as the limit, when $\rho = 0$, of the upper limit of $| R(x, y) |$ in the semicircle of radius ρ, and centre at the point x, has a positive value (which, however, may be indefinitely great) different from zero at every point x of non-uniform convergence; it may be called the measure of the non-uniform convergence at P.* If, in Fig. 3, we divided the semicircle into two quadrants by means of the radius Pp, we may consider separately the upper limits of $| R(x, y) |$ in the quadrants Ppq, Ppq'; when ρ is indefinitely diminished, these upper limits have as limiting values two numbers β_x^+, β_x^-, which may be called the measures of non-uniform convergence at x, on the right and left respectively.

These numbers β_x^+, β_x^- are equivalent to Osgood's indices† of a point $(\bar{b}^+, \bar{b}^-)$, which he defines differently.

If $\beta_x^+ = 0$, $\beta_x^- > 0$, the point x may be said to be one of uniform convergence on the right. If $\beta_x^+ > 0$, $\beta_x^- = 0$, the point x may be said to be a point of non-uniform convergence on the left. At a point of uniform convergence, $\beta_x^+ = \beta_x^- = 0$; β_x is the greater of the two numbers β_x^+, β_x^-.

The number β_x is the saltus of the function $| R(x, y) |$ at the point $(x, 0)$, and this cannot exceed the saltus of $R(x, y)$; now it has been shown that the points at which the saltus of $R(x, y)$ is greater than the arbitrarily assigned number 2σ are points at which the minimum of the function $\lambda_\sigma(x)$ is zero; it has further been shown that these points at which $\lambda_\sigma(x)$ has a zero minimum form a non-dense closed aggregate. It thus appears that those points at which the measure of non-uniform convergence of the series is greater than an arbitrarily assigned positive number form a non-dense closed aggregate.

That there may be points at which the minimum of $\lambda_\sigma(x)$ is zero is in close connexion with the fact which formed the starting point of Osgood's investigation, that the approximation curves whose ordinates represent the function $s_n(x)$ have, in the neighbourhood of points of non-uniform convergence, peaks which remain of finite height, or become of indefinitely great height, as n is indefinitely increased.

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* This term *Grad der ungleichmässigen Convergenz* is employed by Schoenflies (*Bericht über die Mengenlehre*, p. 226), who uses Osgood's definition. The function β_x considered as a function of x is called by Schoenflies the convergency function (*Convergenzfunction*).

† *Loc. cit.*, *American Journal*, p. 166.

8. We proceed now to show that, on the supposition that $s(x)$ is integrable through any interval (x_0, x_1) contained in the interval of convergence of the series, it is a sufficient condition that

$$\int_{x_0}^{x_1} s(x)\,dx$$

is the sum of the series

$$\int_{x_0}^{x_1} u_1(x)\,dx + \int_{x_0}^{x_1} u_2(x)\,dx + \dots + \int_{x_0}^{x_1} u_n(x)\,dx + \dots;$$

that in the interval (x_0, x_1) the measure of non-uniform convergence of the series

$$u_1(x) + u_2(x) + \dots + u_n(x) + \dots$$

is everywhere less than some fixed finite number. Since

$$s(x) = s_n(x) + R_n(x),$$

we have
$$\int_{x_0}^{x_1} s(x)\,dx = \int_{x_0}^{x_1} s_n(x)\,dx + \int_{x_0}^{x_1} R_n(x)\,dx;$$

hence what we have to show is that, under the conditions stated, a number m can be found corresponding to a given number ϵ as small as we please, such that, if $n \geqq m$,

$$\left| \int_{x_0}^{x_1} R_n(x)\,dx \right| < \epsilon.$$

This is equivalent to showing that, corresponding to a given ϵ, a value of y can be found, say y_0, such that

$$\left| \int_{x_0}^{x_1} R(x, y)\,dx \right| < \epsilon,$$

if y has, in the integrand, any fixed value which is $\leqq y_0$.

Take a fixed positive number A; then it has been shown in Art. 7 that the aggregate G of points in the x-axis at which the saltus of $|R(x, y)|$ is greater than A form a closed non-dense aggregate. Now it is well known that such an aggregate G, in the most general case, consists of the end points of an enumerable aggregate of sub-intervals of the given interval (x_0, x_1), together with the limiting points of these end points, and that these sub-intervals are such that no two of them have a common point, except such end points as may be common to two adjacent sub-intervals. These sub-intervals we may suppose to be of lengths $\theta_1, \theta_2, \theta_3, \dots$, where $\theta_r \geqq \theta_{r+1}$; every point of the interval (x_0, x_1) which is not in the *interior* of one of these sub-intervals θ is a point of the aggregate G. If I denote the

content of the aggregate G, then $l-I$, where $l=x_1-x_0$, is the limit of the sum $\theta_1+\theta_2+\theta_3+\dots$; if we take only the first μ of the sub-intervals θ, the points of G all lie in the complementary sub-intervals of l (including the ends), and from the theory of the content a value of μ can be found so great that the sum of the complementary sub-intervals in which the points of G lie is $<I+\epsilon_1$, where ϵ_1 is a fixed arbitrarily small number; for such a value of μ we have

$$\theta_1+\theta_2+\dots+\theta_\mu > l-I-\epsilon_1, \quad \text{and} \quad <l-I.$$

Inside each of the sub-intervals $\theta_1, \theta_2, \dots, \theta_\mu$, we may take a completely enclosed sub-interval θ', such that, if ϵ_2 be a fixed arbitrarily small number,

$$\sum_1^\mu \theta' = \sum_1^\mu \theta - \epsilon_2;$$

then the sum of the μ sub-intervals θ' lies between $l-I-\epsilon_1-\epsilon_2$ and $l-I-\epsilon_2$.

Now, imagine the whole interval l to be divided into $\mu+s$ sub-intervals, μ of which are the above $\theta'_1, \theta'_2, \dots, \theta'_\mu$, and the other s of which are $t_1, t_2, \dots, t_s$, thus

$$l = \sum_1^s t + \sum_1^\mu \theta';$$

all the points of G lie in the sub-intervals t.

First consider $\int R(x, y)\,dx$ taken through the sub-intervals θ'; upon θ'_r as base a rectangle of height $\bar{y}_r$ can be drawn such that the value of $|R(x, y)|$ at any point in the rectangle is $\leqq A+\eta$, where η is a prescribed number as small as we please. For suppose this not to be the case; then there are points of the x-axis in θ'_r such that the fluctuation of $|R(x, y)|$ in areas containing them is $>A$, however small y may be taken, and this is contrary to the hypothesis that at every point of θ'_r the saltus of $|R(x, y)|$ is $\overline{\gtrless} A$; hence $\bar{y}_r$ can be found corresponding to a given η. Let $\bar{y}$ be the greatest of the μ numbers $\bar{y}_1, \bar{y}_2, \dots, \bar{y}_\mu$; then, if $y \overline{\gtrless} \bar{y}$, for every value of x in any of the θ' intervals, we have $|R(x, y)| \leqq A+\eta$; hence the numerical value of $\int R(x, y)\,dx$ taken through all the θ' intervals is

$$\leqq (A+\eta)\sum_1^\mu \theta', \quad \text{or is} \ < (l-I-\epsilon_2)(A+\eta),$$

provided $y \overline{\gtrless} \bar{y}$, and $\bar{y}$, η are so related that they converge to zero together.

Next, consider the s intervals $t_1, t_2, \dots, t_s$, which contain all the points of G. For any point x of G there is a value of y such that,

for it and all smaller values of y, $|R(x, y)| < \sigma$, where σ is a prescribed positive number which we will take to be $< A$: this follows from the fact that $R(x, y)$ is continuous with respect to y at the point $(x, 0)$.

Take a fixed value y_1 of y, and let G_y be the aggregate of those points belonging to G which are such that $|R(x, y)| < \sigma$, when $y \leq y_1$; as y_1 is diminished, G_{y_1} takes in continually more points of G: thus G is the limit of G_{y_1} as y_1 converges to zero.

If I_{y_1} denotes the content of the aggregate G_{y_1}, the points of G_{y_1} may be all included in a finite number of sub-intervals $r_1, r_2, \ldots, r_n$, whose sum Σr is $< I_y + \delta$, where δ is an arbitrarily chosen positive number, the complementary intervals whose sum is $\sum_1^s t - \sum_1^n r$ containing only points of G which do not belong to G_y.

Since there are, by hypothesis, no points in G at which the upper limit of the fluctuation of $R(x, y)$ in (x, y) is not finite, and this upper limit is everywhere less than some fixed finite number, there is a finite upper limit of $|R(x, y)|$ for all values of x which are in the intervals t but not in the intervals r; let B be this upper limit; then the value of $\int R(x, y)\, dx$, taken through those parts of the intervals t which are not in r, is numerically not greater than $B(\Sigma t - \Sigma r)$ or is less than $B(I + \epsilon_1 + \epsilon_2 - I_{y_1})$; it will be observed that B cannot increase as y is diminished.

It now remains to consider the integral taken through the intervals r. Since, by hypothesis, $R(x, y)$, which is equal to $s(x) - s(x, y)$, is integrable in the interval (x_0, x_1), these intervals may be divided into a finite system of sub-intervals, such that the sum of those in which the fluctuation of $R(x, y)$ is greater than or equal to an arbitrarily assigned number is as small as we please. It follows that the intervals r can be further sub-divided in such a way that

$$\Sigma r = \Sigma r' + \Sigma r'',$$

where the r' are intervals in which the fluctuation of $R(x, y)$, for y fixed, is $\geq a$, and the r'' are intervals in which this fluctuation is $< a$, where a is an arbitrarily chosen number, and that this can be so done that $\Sigma r'$ is as small as we please. We shall choose a so small that $a + \sigma < A$. The integral $\int R(x, y)\, dx$ taken through the intervals r' is numerically not greater than $B\Sigma r'$; we have therefore to consider the integral taken through the intervals r'', in each of which the fluctuation is less than a.

Of the intervals r'' some contain points of G_{y_1} and others may not.

Let κ be the sum of the latter; then through them the integral is numerically not greater than κB. For any interval of r'' which contains a point of G_{y_1} the value of $|R(x, y)|$ at every point is less than $\sigma + a$, y having any value $\overline{\gtrless} y_1$; hence the numerical value of $\int R(x, y)\,dx$ taken through these intervals is less than $(\sigma+a)\,\Sigma r''$ or $A\Sigma r''$.

Summing up, we see that, if y is equal to or less than the smaller of the two numbers y_1, $\bar{y}$, the value of $\left|\int_{x_0}^{x_1} R(x, y)\,dx\right|$ is less than

$$(l-I-\epsilon_2)(A+\eta)+B\,(I-I_{y_1}+\epsilon_1+\epsilon_2)+B\,(\Sigma r'+\kappa)+A\Sigma r''.$$

If we suppose A, y_1, $\bar{y}$ to have fixed values, the number ϵ_2 can be chosen to be as small as we please; hence, if y_0 be the smaller of the two numbers y_1, $\bar{y}$, we see that the value of $\left|\int_{x_0}^{x_1} R(x, y)\,dx\right|$ for $y \overline{\gtrless} y_0$ is less than

$$(A+\eta)(l-I+\Sigma r'')+B\,(I-I_{y_1}+\epsilon_1)+B\,(\Sigma r'+\kappa),$$

$\Sigma r'$ is as small as we please to make it; hence the integral is less than

$$(A+\eta)(l-I+\Sigma r)+B\,(I-I_{y_1}+\epsilon_1)+B\kappa$$

or than

$$(A+\eta)(2l-I)+B\,(I-I_{y_1}+\epsilon_1)+B\kappa.$$

It will be proved below that I is the limit, when y_1 converges to zero, of the content I_{y_1}; assuming this for the present, y_1 can be chosen so small that $I-I_{y_1} < \lambda$, where λ is an arbitrarily chosen positive number, we have also $\kappa < \lambda$; thus the absolute value of the integral is less than

$$(A+\eta)\,2l+B\,(2\lambda+\epsilon_1).$$

Let

$$\epsilon = p+q+r+s,$$

where p, q, r, s are all positive; choose $A < \dfrac{p}{2l}$; then choose $\bar{y}$ so that $\eta < \dfrac{q}{2l}$; then choose y_1 so that $2B\lambda < r$; and, lastly, take the number of θ intervals such that $B\epsilon_1 < s$. A value y_0 has thus been found of y such that $\left|\int_{x_0}^{x_1} R(x, y)\,dx\right| < \epsilon$, provided $y \overline{\gtrless} y_0$.

9. It only remains for us to prove the theorem which we have assumed in § 8, that the limit of I_{y_1}, as y_1 converges to zero, is I. The proof here given is substantially identical with the proof given by Osgood (*loc. cit.*), to whom the theorem is due, and who pointed out

that the truth of the theorem essentially depends upon the aggregate G being a closed one.

As y_1 diminishes, I_{y_1} can never diminish, and it can clearly not be greater than the positive number I; hence, by a well-known theorem, I_{y_1} has a limit which is $\leqq I$. The aggregate G_{y_1} can be enclosed in a finite number of sub-intervals, whose sum $\Sigma\phi^{(y_1)}$ is less than $I_{y_1}+\delta_{y_1}$, where δ_{y_1} is an arbitrarily chosen positive number. Next take a value y_2 of y, $<y_1$; corresponding to y_2 we have an aggregate G_{y_2} whose content I_{y_2} is not less than I_{y_1}. Those points of G_{y_2} which are not in the intervals $\phi^{(y_1)}$ can be enclosed in a finite number of further intervals $\phi^{(y_2)}$, of which the extremities are not points of G, and so that

$$\Sigma\phi^{(y_1)}+\Sigma\phi^{(y_2)} < I_{y_2}+\delta_{y_1}+\delta_{y_2},$$

where δ_{y_2} is an arbitrarily chosen positive number. If we apply this process to the aggregates $G_{y_1}, G_{y_2}, \ldots, G_{y_n}, \ldots$, corresponding to a sequence $y_1, y_2, \ldots, y_n, \ldots$ of values of y, which converges to zero, we have a finite number of intervals which enclose the points of G_{y_n}, and whose extremities are not points of G, such that

$$\Sigma\phi^{(y_1)}+\Sigma\phi^{(y_2)}+\ldots+\Sigma\phi^{(y_n)} < I_{y_n}+\delta_{y_1}+\delta_{y_2}+\ldots+\delta_{y_n} < I_{y_n}+\delta,$$

if $\delta_{y_1}, \delta_{y_2}, \ldots$ be taken to be a decreasing sequence of numbers whose sum converges to a given arbitrarily chosen number δ.

As n increases indefinitely, the number of intervals r cannot increase indefinitely; for, if it did so, the extremities of these intervals must have one or more limiting points x', and it would be possible, out of the intervals whose extremities have x' as limiting point, to choose an aggregate of points of G which would have x' as limiting point; hence, as G is a closed aggregate, x' would itself be a point of G, and this is impossible, as x' is not in any of the intervals. Thus, as n increases, the number of intervals ϕ reaches a maximum number N which is never exceeded. These N intervals contain in their interior all the points of G; hence all the points of G are enclosed within M intervals, whose sum is $< \mathrm{L}_{y=0}\, I_y+\delta$; if now $\mathrm{L}\, I_y$ were less than I, we could choose δ so that the sum of these intervals was less than I, and it is impossible that all the points of an aggregate G can be enclosed in a finite number of sub-intervals whose sum is less than the content of G; thus we have proved that $\mathrm{L}_{y=0}\, I_y = I$.

The theorem has now been fully established that the integral of the sum of a convergent series of which every term is a continuous function is equivalent to the sum of the integrals of the separate terms, provided the sum of the series is integrable throughout the

interval, and provided further the measure of non-uniform convergence at every point of the interval is less than some fixed number.

If the measure of non-uniform convergence is at no point of the interval of integration indefinitely great, then this measure is for every point of the interval less than some fixed finite number. For, since those points at which the measure of non-uniform convergence is greater than a fixed number form a closed aggregate, if for any sequence of points this measure continually increased and had no finite upper limit, for a limiting point of the sequence the measure would be indefinitely great, which is contrary to hypothesis.

Networks. *By* SAMUEL ROBERTS. Read and received January 9th, 1902.

1. This paper treats of certain networks, (1) with triangular meshes, (2) with polygonal meshes. These will, for shortness, be called respectively triangular and polygonal networks. They are, of course, intimately connected with the problem of colouring maps with four colours only.

The doubts and difficulties which have arisen with regard to the demonstration of the general theorems involving the solution of the problem in question show the expediency of discussing limited and defined cases, and passing to more general results step by step. The subject, in fact, is larger and more intricate than the simplicity of the empirical solution would lead one to expect.

2. Consider a piece of triangular network the sides of whose meshes are rectilinear, as represented concretely in Fig. 1. The outer contour is supposed to be continuous, and consists of not less than three sides. The connectors (*i.e.*, the sides of the meshes) do not meet except at knots. These are variously multiple, and the number of connectors meeting in a knot is the weight of the knot. Any two knots are joined by one connector only: this is implied by the triangular and rectilinear character of the meshes. The case of

networks united by a single knot is excluded for a reason which will presently appear.

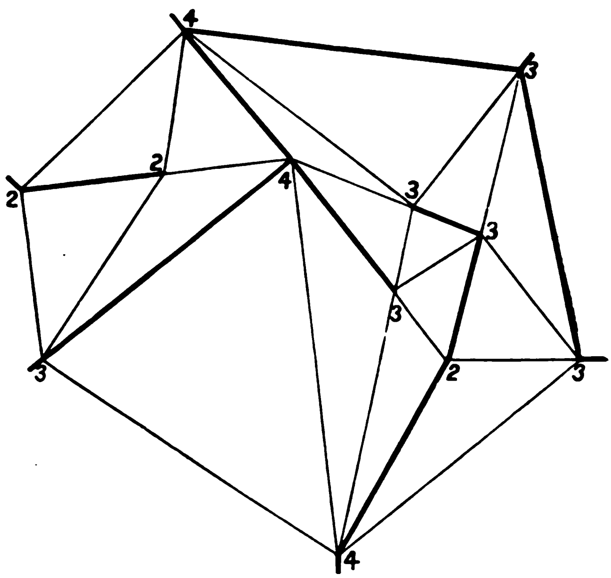

FIG. 1.

3. When the outer contour is made up of more than three sides the network is said to be incomplete; when the contour consists of three sides the network is considered as complete.

Generally it is desirable to add another knot to an incomplete network, so as to make it complete by joining the added knot to the knots of contour by connectors. These may be made rectilinear by suitable adjustment, as in Fig. 2, which gives the completion of Fig. 1.

4. Incomplete networks united by a single knot were excluded, because in the complete diagram the added knot would be joined to the knot of junction by two connectors, both of which could not be rectilinear. As a matter of practical convenience it is plain that in drawing diagrams the actual straightness of the connectors is immaterial, if they can be made straight by adjustment and do not deceive the eye. Unessential alterations may be made in the dimensions and directions of the connectors, and the arrangement of the knots.

In normal networks, knots of less weight than three are not admitted. Networks united by a single bond are at present excluded, although the completed diagram would be triangular. Such forms imply multiple connectors.

5. Some general and important numerical theorems are readily obtained by regarding a complete triangular network as the projection on a plane of a polyhedron with triangular faces only. For, if S, E, F are respectively the numbers of the solid angles, the edges, and the faces of such a polyhedron, Euler's formula gives

$$S-E+F = 2.$$

Now let s, e, f, and w be the numbers of the knots, the connectors, the plane triangles (including the base), and the weight of the plane system; then

$$w = 2e = 3f;$$

and, if we take $n+2$ knots to form the network,

$$s = n+2, \quad e = 3n, \quad f = 2n, \quad w = 6n.$$

6. In numerous concrete cases, of which Fig. 2 affords a simple example, two separated branched forms or trees can be determined,

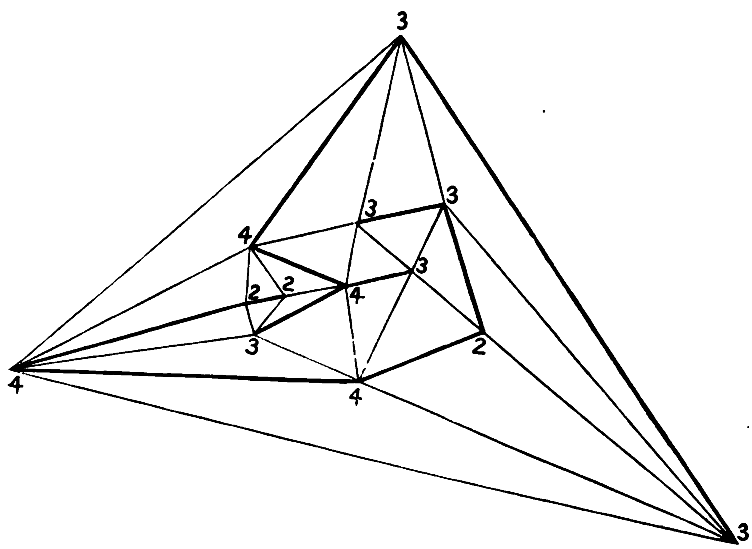

Fig. 2.

passing through all the knots, not returning on themselves, so as to

form loops, and such that no two knots on the same tree are joined by a simple connector. The trees are connected by $2n$ bonds,

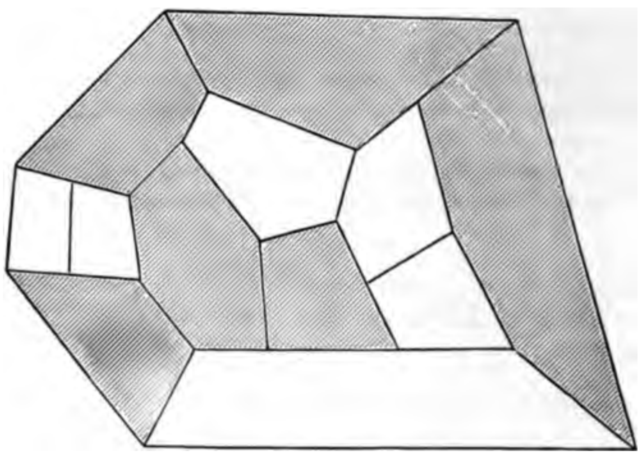

FIG. 3.

i.e., simple connectors which do not intersect except at knots, or at all events can be modified so as not to do so. In the general case of $n+2$ knots, the number of branches taken together is n.

Let p be the number of branches on one tree, and denote by $a_1, a_2, \ldots, a_{p+1}$ the knots thereon. The characters of the trees which can be constructed from the p branches are indicated by the product $(a_1+a_2+\ldots+a_{p+1})^{p-1}(a_1, a_2, \ldots, a_{p+1})$ (Cayley. "Theorem on Trees," *Quart. Jour.*, Vol. XXIII., p. 376).

The arguments in the expansion show the several characters of possible trees not depending on the lengths or directions of the branches, the multiplicity of the knots being indicated by the powers of their symbols, in each term. The sum of the indices is $2p$.

The gross weight of the knots of the tree (*i.e.*, taking account of the weight due to the $2n$ simple connectors) is $2(n+p)$. If therefore we deduct two from the weight of each, we get

$$2(n+p)-2(p+1) = 2(n-1),$$

showing the weight (less two) due to the $2n$ simple connectors.

Let q be the number of branches on the second tree; the net weight is still $2(n-1)$, due to the simple connectors.

7. For two given trees, it is plain the $2n$ simple connectors can be variously arranged and will produce different complete diagrams.

If now we suppose p and q to vary integrally, and so that

$$p+q=n,$$

and take up all possible positions of the simple connectors as well as of the branches, we arrive implicitly at least at the totality of complete, normal, triangular networks of the order $n-2$, which, as we may say, are subject to solution by trees, *i.e.*, have an open solution.

We must at present exclude the uses of p or $q=0$, because multiple connectors are then introduced. The case is, however, considered in § 16.

8. To fix the ideas, it is useful to dwell on a concrete example of a complete normal triangular network, such as Fig. 2, where the branches are represented by thickened lines. It is, however, easy to draw a more complicated diagram. We observe that the correlative trees are so far dependent on one another that for a given network one determines the other; for the branches of the two together pass through all the knots.

We see also that, if we give a zero value to any branch, thus causing two simple connectors on each side of the branch to coalesce, we lose practically two connectors and one branch, while the reduced figure of $n+1$ knots has a solution consisting of the tree from which one branch has been removed and the correlative tree unchanged. The weight of the new knot is less by two than the weight of the component knots, and the weight of the knots on the correlative tree is also reduced by two.

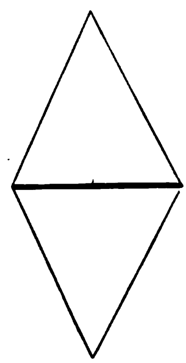

Fig. 4.

Thus the whole figure is made up of quadrilateral elements such as Fig. 4 represents, simple connectors forming the sides and officiating twice as boundaries. In the general case of $n+2$ knots there are p such elements making up the one tree and q such making up the correlative tree, and $p+q=n$. The $2n$ triangles are thus accounted for.

9. In order to pass from a diagram of $n+1$ to one of $n+2$ knots, retaining a tree solution of the lower form, we have to work with one branch and two simple connectors. The only admissible way of adding two new triangles, while preserving continuity of solution

and increasing the order by unity, is to divide and open out a simple connector on each side of a knot so as to separate the knot into two components; the net weight of the two components will be two more than that of the original knot; the original simple connectors and the additional ones will intersect at knots on the correlative tree. Thus one or the other of the trees of the lower form receives an addition of one branch, and the net weight of each tree is increased by two. This process is, of course, applicable to all the tree solutions of a complete triangular network of $n+1$ knots.

So far we have considered the totality of complete normal triangular networks soluble by a correlative pair of trees the sum of whose branches is n in number. But for the purpose of induction something more is required. We make, then, here the postulate that the totality of networks in question is really the totality of all such networks of the order $n+2$ which are normal, as not having multiple connectors.

10. In considering tree solutions, it is convenient to say that knots on the same tree are congruent, and knots on different trees are incongruent.

Let us take now an arbitrary, complete, normal triangular network of $n+3$ knots, $2n+2$ meshes, $3n+3$ connectors, &c. Extract from the graph a fragment such as Fig. 5, bearing in mind that the network may be referred to any mesh as base. Fig. 5 consists of a

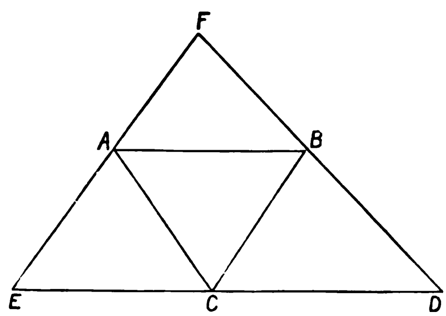

Fig. 5.

mesh ABC with three others each having a common side with ABC, viz., AFB, BDC, CEA. Unite the knots A and B in (AB). If the reduced form of $n+2$ knots (which by the postulate is susceptible of a tree solution in one or more ways) possesses such a solution, in which F, C are congruent with one another, but incongruent with

(AB), we can open out again the united knot and thus get a tree solution, the tree through F and C being unchanged except that the net weight of the knots thereon is increased by two, and one branch is added to the correlative tree. But we may suppose that no solution of the reduced form satisfies the condition; then AB must be a simple connector and A, B incongruent.

It is important to remark that, if AB is a branch, we cannot compatibly suppose that BC is also a branch. In fact the tree to which AB is a branch passes through A and B, but not through C and F; whereas the second tree passes through B and C, but not through A and D. Hence the supposition that AB and BC may be branches refers only to two different solutions.

But we may suppose that AB is not a branch, the condition failing, and that BC does satisfy the condition, viz., that when $BC = 0$ the second reduced form has a tree solution making the united point (BC) incongruent with A and D, so that a tree solution of the higher form is obtained by opening out again (BC).

But we may again suppose that the condition fails, in which case BC also is a simple connector, and A is congruent with C. Then EA and EC are simple connectors.

But by the postulate the reduced form when $AC = 0$ has a tree solution. If this makes the united point (AC) incongruent with B and E, there is a tree solution of the higher form; but, if it be that there is no such solution, AC is a simple connector, and A is incongruent with C. In that case our original suppositions that AB, BC are simple connectors fail. Both suppositions, however, cannot fail, for then AB, BC would be branches of the same tree, which we have seen cannot be. Therefore one only fails, and either AB or BC is a branch, giving a solution of the higher form.

The argument may be put in different forms, but they amount to very much the same thing. Take the form in Fig. 4. Assume that the four sides of the quadrilateral are simple connectors. This is possibly a compatible form when the diagonal is a branch. If so, contract the diagonal to zero; the united point is incongruent with the opposite vertices, and there is a solution of the reduced form which may be extended to give a solution of the higher form. But it may be that the supposed simple connectors are incompatible. In that case, there must be a break in consequence of the reduced form, when one of the sides is made zero, fulfilling the condition of a branch. There will, in fact, be two such breaks to make the form compatible. We may have two opposite side branches, or two

adjacent sides. Cases are known in which a particular bond cannot be a branch at all, and also in which two intersecting connectors (with a third intervening between them) cannot be branches at all. The intervening bond may sometimes be either a simple connector or a branch.

The whole matter depends on the congruity or otherwise of two apices of a triangle, and this applies to any decompositions of a triangular graph, so that each triangle has an apex incongruent to the other two, but in this case we may have decompositions into odd polygons, &c., not giving a solution in our present sense.

Of course a bond may be a branch springing from a knot and not continued. Since our given graph is supposed to be an arbitrary complete normal triangular network, we are now justified in stating that any such form normal and of an order greater than three is soluble by trees.

11. When a tree solution of a normal complete triangular network is found, we can draw a continuous line cutting all the simple connectors, but not the trees. Hence, if we suppose the tree branches coloured green, we may colour alternate simple connectors blue, and the rest red. In this way, the meshes of the form become coloured each side differently, only three colours being employed.

12. A normal complete triangular network can be derived from a corresponding polygonal network by an obvious process. The meshes of the latter form are subject only to the condition that no two polygons have more than one common side. The converse is also true. To fix the ideas the diagram of Fig. 3 is derived from that of Fig. 2. It is, in fact, a kind of reciprocal form in which a polygonal mesh of μ sides corresponds to a knot of weight μ in Fig. 2, and a mesh of Fig. 2 corresponds to a triple knot of Fig. 3. In the general case the $n+2$ knots of the triangular network correspond to the $n+2$ polygons of the new form, the outer contour being reckoned in; the $2n$ triangles of one form correspond to the $2n$ knots of the other. The branches of a tree solution of the triangular form indicate the connectors of the polygonal form which must be cut away in order to obtain a closed polygon of $2n$ sides, thus giving a solution of the form.

We now see the inward meaning of the separate trees of the solution of the triangular form; for it will be seen that the polygon

of $2n$ sides is equally determined whether we clear of obstructing connectors the outside or the inside of the continuous contour.

Further, we can evidently colour the three sides meeting in each triple knot with three different colours, three colours only being used. This is, of course, a particular case of the late Prof. Tait's theorem. From it we conclude that the polygons can be coloured so that no two adjacent polygons shall be of the same colour, four colours only being employed. This is usually shown as follows.

Let A, B, C represent the three colours distinguishing the connectors meeting in a triple knot, and let a, b, c, d be the four colours at our disposal, and suppose A is equivalent to ab and cd, B to ac and bd, and C to ad and bc. The alternative values are sufficient for the end in view, and it is hardly necessary to enter into details.*

This conclusion covers the case of a block of countries, no two having more than one frontier in common with a neighbour, and no country extending from sea to sea. If more bonds than three meet in a knot, we may draw a small circle about the knot, colour the diagram and then contract the circle to zero radius. For geographical language applicable to the subject see Lucas, *Récréations Mathématiques*, t. IV.

13. It would appear that Tait's theorem has been enunciated in too general terms. Prof. Petersen, of Copenhagen, using a special terminology of his own, states it thus: "Un graphe du troisième degré, qui n'a pas de feuilles, peut être décomposé, en trois du premier degré." Here *feuille* means a part of the *graphe* which is joined to the rest by one bond only, and *degré* is the number (the same throughout) of the bonds meeting in each knot. The Professor, later in the same communication, gives an example in which the theorem does not hold (*L'Intermédiaire des Mathématiciens*, Vol. v., p. 226). It will be sufficient to give his diagram, Fig. 6, which speaks for itself. The connectors necessarily cross and cannot be unravelled; if we attempt to derive a triangular network, we fail, and are involved in a maze of multiple lines.

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* Lucas, *Récréations Mathématiques*, t. IV., p. 193. **Process also given by Tait and others, I believe.**

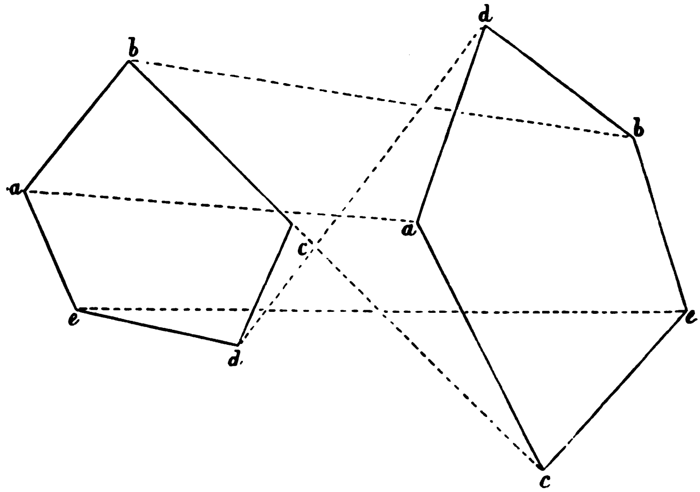

Fig. 6.

14. But I am not sure that the late Prof. Tait was responsible for such an unguarded statement of his theorem. For, I find in his address on Listing's *Topologie* (*Phil. Mag.*, Vol. xvii., pp. 30 *et seq.*) the following words :—" But, if $2n$ points in a plane be joined by $3n$ lines no two of which intersect, so that every point is a terminal of three different lines, the figure requires n separate pen strokes. It has been shown that in this case (unless the points be divided into two groups, between which there is but one connecting line) the $3n$ lines may be divided into three groups of n each, such that one of each group ends at each of the $2n$ points."

The condition that no two of the lines " intersect " seems to mean " cross," since three intersect at each point.

Be this as it may, Prof. Petersen's example sufficiently shows that the theorem in question is not absolutely general. Nevertheless, the unsafe extension was natural, since we can in many ways cause the bonds to cross, without affecting the truth of the theorem in particular cases.*

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* If we take several even polygons and complete the graph of the third degree by connectors, these may be coloured the same, and may therefore be arranged in all possible ways.

15. In the present state of the literature of the subject, it was convenient to limit the research considerably. It has been found that the open solution by means of trees is possible in normal triangular networks. We are thus enabled to colour the three sides of the meshes severally with three different colours, three colours only being used. In such cases, the sets of connectors of one colour do not often give three open solutions, but also looped solutions made up of closed circuits, partial trees, and knots of even weight. The number of open solutions may be 3, 2, or 1, and there may be two looped solutions or one such solution. As a matter of fact, the meshes may sometimes be coloured with three colours in such a way that all three sets of connectors may give loops, or even knots.

The great variety of forms which looped solutions assume presents a wide field of investigation, but little worked. I can only here make a few remarks which are near the surface of the subject. Such a solution consists of two correlative sets of elements, the defining bonds of which are branches, the elements only connected by simple connectors. The sum of the weights of the knots of each element is even; so also is the number of proper sides of a loop; isolated knots are even; there may be trees separate or attached to the sides of loops, and these may have sides in common or be united by knots.

Thus let $2\mu_1, 2\mu_2, \ldots$ be the gross weight of the knots on the elements of one of the correlative systems; and let $p_1, p_2, \ldots$ be the numbers of branches on the respective elements; then we have

$$\Sigma 2\mu - \Sigma 2p = 2n,$$

and for the other member of the system, if $2r_1, 2r_2, \ldots$ and $q_1, q_2, \ldots$ are the corresponding numbers, we have

$$\Sigma 2r - \Sigma 2q = 2n,$$

$$\Sigma p + \Sigma q = n.$$

Under these conditions, the looped solution enables us to colour the sides of each mesh differently, three colours only being used. A few figures easily drawn will make the matter clearer than description, always inadequate. A simple example is Fig. 2. Here we find one open solution and two looped solutions. The one, it is easy to see, consists of three groups, two of them giving

$$18+4 = 22,$$

and the remaining group giving also 22. In this $n = 11$. The other

looped solution has four groups, of which three give

$$10+8+4 = 22,$$

and the remaining group gives 22.

I must pass to a subject more relevant to the purpose of this paper. An important difference between open solutions and looped ones is worth stating, namely, that giving a zero value to a branch forming a boundary of a loop will not give a solution of the lower form, because the resulting loop will be odd.

16. We passed over previously the case in which one tree of an open solution degenerates into a knot, and the correlative tree has n branches. We are then introduced to multiple connectors. Fig. 7

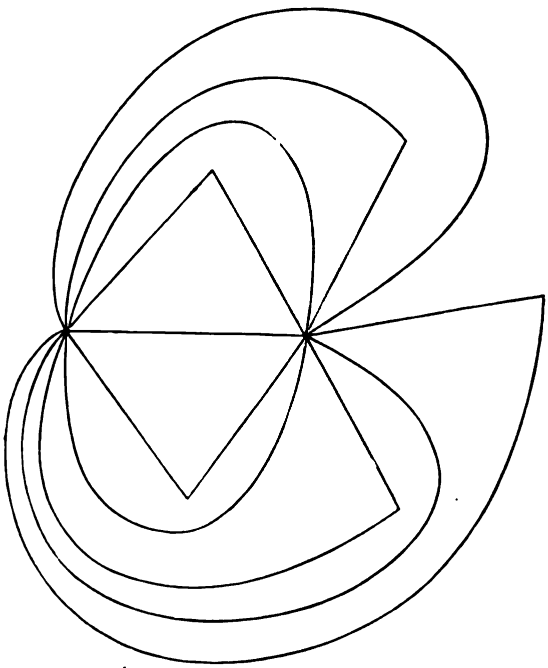

Fig. 7.

gives an example of the corresponding complete form, while Fig. 8 shows the correlative polygonal network. Generally a multiple bond on a complete triangular network corresponds to common sides of equal multiplicity in the associated polygonal network. Our dis-

cussion is limited to triangular systems and associated polygonal

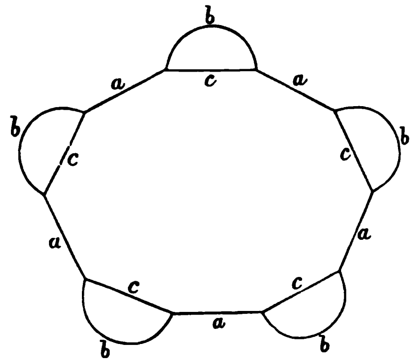

Fig. 8.

ones. Therefore each pair of connectors must contain a sub-form which completes the meshes.

Fig. 9 represents a simple case, and Fig. 10 gives the associated

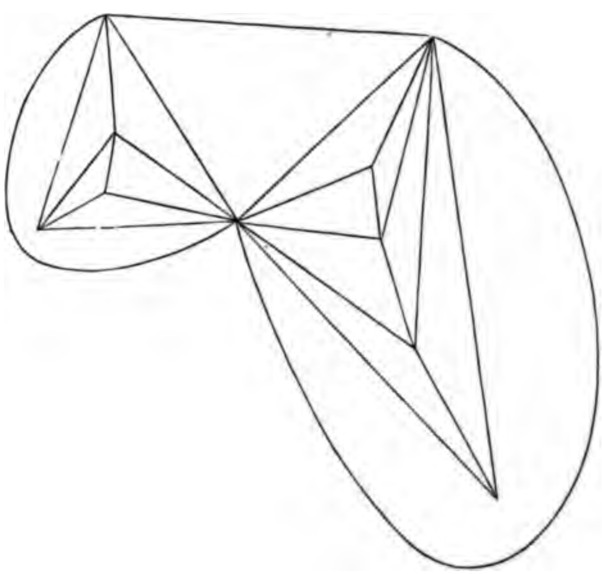

Fig. 9.

polygonal network. The contour, of course, corresponds to the knot of weight 11. The thickened lines give an open solution. The

polygonal figure is also given in Fig. 11, the contour corresponding to a knot of weight 5.

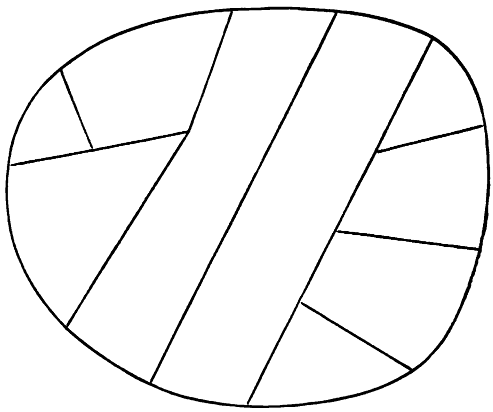

FIG. 10.

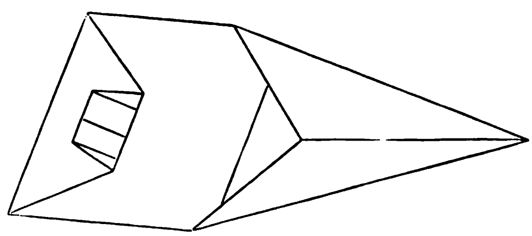

FIG. 11.

17. It is plain that in the colouring process the multiple connectors will be of one colour. In fact, when we have coloured a complete triangular network, we can add a bond joining any two of the apices of the base so as to leave the whole form bounded by two bonds, which must evidently receive the same colour. The same thing follows from the consideration that in the polygonal form associated with the triangular one we must remove all or none of the common sides of two adjacent meshes.

A little consideration will convince us that, if we substitute in a complete network with multiple connectors simple connectors and find an open or a looped solution, we can get a solution of the unmodified form by replacing the multiple connectors and their enclosed forms and harmonizing the colours.

It is easy to find triangular networks with multiple connectors not susceptible of open solutions: such a case is represented by Fig. 12, and the correlative polygonal network is given by Fig. 13.

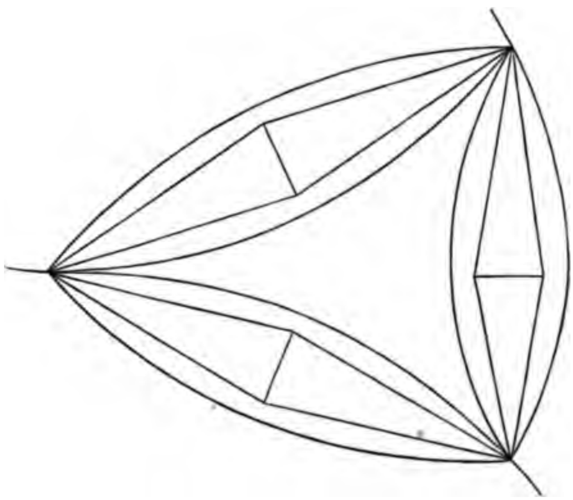

Fig. 12.

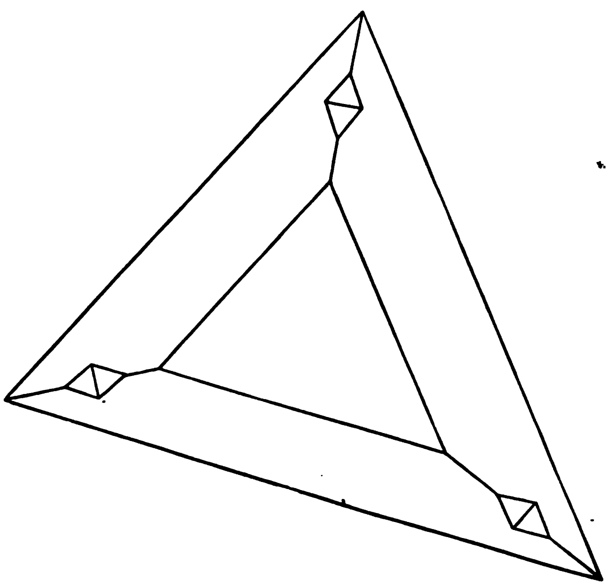

Fig. 13.

We are now in a position to colour maps where adjacent countries have separated frontiers. Islands, enclaves, &c., present no difficulty. More details as to sundry corollaries deduced in anticipation of demonstration of theorems to which they belong will be found in the *American Journal of Mathematics* and elsewhere.

Thursday, February 13th, 1902.

Dr. HOBSON, F.R.S., President, in the Chair.

Thirteen members present.

Prof. Lamb read a paper "On Boussinesq's Problem." Messrs. Love, Hargreaves, Cunningham, Macdonald, and the President, took part in the ensuing discussion.

Mr. A. Young read a second paper "On Quantitative Substitutional Analysis."

Prof. Love explained a new proof of a well-known theorem concerning zonal harmonics.

The following papers were communicated by the President:—

(i.) On the Density of Linear Sets of Points, (ii.) On Closed Sets of Points defined as the Limit of a Sequence of Sets of Points: Mr. W. H. Young.

On Plane Cubics: Prof. A. C. Dixon.

On the Wave Surface of a Dynamical Medium, Æolotropic in all respects: Prof. Bromwich.

Elementary Proof of a Theorem for Functions of several Variables: Dr. H. F. Baker.

The following presents were made to the Library:—

"Educational Times," February, 1902.

"Indian Engineering," Vol. xxx., Nos. 25, 26; Vol. xxxi., Nos. 1–3; Dec. 21–Jan. 18, 1901–1902.

"Supplemento al Periodico di Matematica," Anno v., Fasc. 3; Livorno, 1902.

Société des Naturalistes de Varsovie:—"Procès-Verbal," 1900; "Comptes Rendus," 1899–1900.

"Revista Trimestral de Matemáticas," Año 1, Núm. 4; Zaragoza, 1901.

"Annals of Mathematics," Series 2, Vol. III., No. 2; 1902.

Dini, U.—"Sopra una Classe di Equazioni a derivate parziali di second' Ordine," 4to; Roma, 1901.

"Nautical Almanac Appendix," 1902.

"Report of the Superintendent of the United States Naval Observatory," ending June 30, 1901, 8vo; Washington, 1901.

"Württemberg-Mitteilungen," January, 1902.

Dickson, Prof. L. E.—

"The Configurations of the 27 Lines on a Cubic Surface, and the 28 Bitangents to a Quartic Curve," pamphlet, 8vo; New York, 1901.

"The Groups of Steiner in Problems of Contact," pamphlet, large 8vo; New York, 1902.

"Representation of Linear Groups as Transitive Substitution Groups," 4to pamphlet.

The following exchanges were received :—

"Proceedings of the Royal Society," Vol. LXIX., Nos. 454, 455; 1902.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXVI., Hefte 1, 2; Leipzig, 1902.

"Bulletin de la Société Mathématique de France," Tome XXIX., Fasc. 4; Paris, 1901.

"Transactions of the American Mathematical Society," Vol. III., No. 1; New York, 1902.

"Bulletin of the American Mathematical Society," Series 2, Vol. VIII., No. 4; New York, 1902.

"Monatshefte für Mathematik und Physik," Jahrgang XIII., Vierteljahrschrift 1, 2; Wien, 1902.

"Bulletin des Sciences Mathématiques," Tome XXV., Dec.; Paris, 1901.

"Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. VII., Fasc. 12; Napoli, 1901.

"Journal für die reine und angewandte Mathematik," Bd. CXXIV., Heft 2; Berlin, 1901.

"Annali di Matematica," Serie 3, Tomo VII., Fasc. 1; Milano, 1902.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. X., Fasc. 12; Vol. XI., Fasc. 1, 2; Roma, 1901–1902.

"Revue Semestrielle des Publications Mathématiques," Tome X., Pte. 1; 1902.

"Journal of the Institute of Actuaries," Vol. XXXVI., Pt. 4; 1902.

"Nieuw Archiev voor Wiskunde," Reeks 2, Deel V., St. 3; Amsterdam, 1901.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," Nos. 39–53; 1901.

"Wiadomósci Matematyczne," Tom V., Zeszyt 4–6; Warsaw, 1901.

"Proceedings of the Cambridge Philosophical Society," Vol. XI., Pt. 4; 1902.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Math.-Phys. Klasse, Heft 2; 1901.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXX., Jahrgang 1899, Heft 3; Berlin, 1901.

During the year 1901 the following presents to the Library have been sent, in the first instance, to Prof. Love, to be indexed for the "International Catalogue of Scientific Literature":—

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. XLV., Parts 1-4; Vol. XLVI., Part 1.

"Proceedings of the Edinburgh Mathematical Society," Vol. XIX.

"Transactions of the Institution of Naval Architects—Memoirs of the Spring and Summer Meetings of 1901."

"Transactions of the Institution of Engineers and Shipbuilders in Scotland," Vol. XLIV., Parts 1-3.

"Journal of the Institute of Actuaries," Vol. XXXV., Part 6; Vol XXXVI., Parts 2, 3.

"Transactions of the Insurance and Actuarial Society of Glasgow," Ser. 5, Nos. 1-6.

Copies of the following were also sent especially for the purpose of the Catalogue:—

"The Mathematical Gazette," Vol. II., No. 1 and Nos. 26-30.

"The Educational Times," Vol. LIV., Nos. 477-488.

"Journal of the Royal Statistical Society," Vol. LXIV., Parts 1-3.

On Boussinesq's Problem. *By* HORACE LAMB.

Read and received February 13th, 1902.

The particular problem here referred to is that of finding the displacements produced in a semi-infinite isotropic solid by pressures applied normally to the plane boundary. This was first solved by Boussinesq,* independent investigations have been given by Hertz† and Cerruti,‡ and quite recently a very ingenious solution has been published by Prof. Michell in the Society's *Proceedings*.§ It may appear that there is hardly room for further discussion of the sub-

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* In the *Comptes Rendus*, Vols. LXXXVI.-LXXXVIII. (1878-9); see also his book *Applications des Potentiels*, &c., Paris, 1885.

† *Crelle*, Vol. XCII. (1881); reprinted in *Werke*, Leipzig, 1895, Vol. I., p. 155.

‡ *R. Accad. dei Lincei, Mem. fis. mat.*, t. XIII. (1882). An account of Boussinesq's and Cerruti's investigations, which include also the effect of tangential stresses on the surface, is given in Love's *Elasticity*, Vol. I., p. 248.

§ Vol. XXXI., p. 183 (1899).

ject, but the following method is perhaps worth notice, as being perfectly simple and straightforward, and requiring only the knowledge of one or two integral properties of Bessel's functions. It is suggested by H. Weber's method of treating various potential problems.*

The solid is supposed to be bounded by the plane $z = 0$, and to extend to infinity on the side for which $z > 0$. We aim first at finding the effect of a distribution of surface pressure which is symmetrical about the origin, but otherwise arbitrary.

In a usual notation, the Cartesian equations to be satisfied in the interior of the solid are

$$\left.\begin{aligned} \nabla^2 u = -\frac{\lambda+\mu}{\mu}\frac{d\theta}{dx}, \quad \nabla^2 v = -\frac{\lambda+\mu}{\mu}\frac{d\theta}{dy}, \quad \nabla^2 w = -\frac{\lambda+\mu}{\mu}\frac{d\theta}{dz} \\ \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} = \theta \end{aligned}\right\} . \qquad (1)$$

Since there is symmetry about Oz, we introduce cylindrical coordinates, and write

$$x = \varpi\cos\omega, \quad y = \varpi\sin\omega, \quad u = q\cos\omega, \quad v = q\sin\omega.$$

The equation $\nabla^2\theta = 0$, which is implied in (1), then takes the form

$$\frac{d^2\theta}{d\varpi^2}+\frac{1}{\varpi}\frac{d\theta}{d\varpi}+\frac{d^2\theta}{dz^2} = 0;$$

this is satisfied by

$$\theta = Ae^{-mz}J_0(m\varpi), \qquad (3)$$

where m is supposed positive in order to secure finiteness for $z = \infty$. To find the corresponding values of q and w we have

$$\left.\begin{aligned} \frac{d^2w}{d\varpi^2}+\frac{1}{\varpi}\frac{dw}{d\varpi}+\frac{d^2w}{dz^2} = \frac{\lambda+\mu}{\mu}Ame^{-mz}J_0(m\varpi) \\ \frac{d^2q}{d\varpi^2}+\frac{1}{\varpi}\frac{dq}{d\varpi}-\frac{q}{\varpi^2}+\frac{d^2q}{dz^2} = -\frac{\lambda+\mu}{\mu}Ame^{-mz}J_0'(m\varpi) \end{aligned}\right\} . \qquad (4)$$

If we assume that, as regards dependence on ϖ,

$$w \propto J_0(m\varpi), \quad q \propto J_1(m\varpi),$$

* See Gray and Mathews, *Bessel Functions*, or H. Weber, *Part. Diff.-Gleichungen d. math. Physik*, Brunswick, 1900–01. The second volume of the latter work has just appeared; it contains yet another solution of Boussinesq's problem, p. 188.

we have
$$\left.\begin{aligned}\frac{d^2w}{dz^2}-m^2w &= \frac{\lambda+\mu}{\mu}Ame^{-mz}\\ \frac{d^2q}{dz^2}-m^2q &= \frac{\lambda+\mu}{\mu}Ame^{-mz}\end{aligned}\right\}, \tag{5}$$
where the factors involving ϖ are omitted. Hence we obtain
$$\left.\begin{aligned}w &= \left(-\frac{\lambda+\mu}{2\mu}Az+\frac{B}{m}\right)e^{-mz}J_0(m\varpi)\\ q &= \left(-\frac{\lambda+\mu}{2\mu}Az+\frac{C}{m}\right)e^{-mz}J_1(m\varpi)\end{aligned}\right\}. \tag{6}$$
Since
$$\theta = \frac{dq}{d\varpi}+\frac{q}{\varpi}+\frac{dw}{dz}, \tag{7}$$
the constants A, B, C are not independent, being connected by the relation
$$B-C = -\frac{\lambda+3\mu}{2\mu}A. \tag{8}$$
Subject to this condition, the formulæ (6) constitute a typical solution of our equations (1), in the case of symmetry. The corresponding values of the surface stresses are
$$\left.\begin{aligned}p_{zz} &= \lambda\theta+2\mu\frac{dw}{dz} = -\mu(A+2B)J_0(m\varpi)\\ p_{z\varpi} &= \mu\left(\frac{dq}{dz}+\frac{dw}{d\varpi}\right) = -\left\{\tfrac{1}{2}(\lambda+\mu)A+\mu(B+C)\right\}J_1(m\varpi)\end{aligned}\right\}, \tag{9}$$
where z has been put $= 0$ after the differentiations. Hence (6) will give the displacements produced by prescribed surface stresses of the types
$$p_{zz} = PJ_0(m\varpi),\quad p_{z\varpi} = QJ_1(m\varpi), \tag{10}$$
provided
$$A+2B = -\frac{P}{\mu},\quad \frac{\lambda+\mu}{2\mu}A+B+C = -\frac{Q}{\mu}. \tag{11}$$
Hence, and from (8), we find
$$\left.\begin{aligned}\theta &= \frac{P-Q}{\lambda+\mu}e^{-mz}J_0(m\varpi),\\ w &= \left[-\frac{P-Q}{2\mu}z+\frac{1}{m}\left\{-\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}P+\frac{Q}{2(\lambda+\mu)}\right\}\right]e^{-mz}J_0(m\varpi)\\ q &= \left[-\frac{P-Q}{2\mu}z+\frac{1}{m}\left\{\frac{P}{2(\lambda+\mu)}-\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}Q\right\}\right]e^{-mz}J_1(m\varpi)\end{aligned}\right\}. \tag{12}$$

Since the hypothesis of a symmetrical *tangential* surface traction does not lead to anything very interesting, we now put $Q = 0$. On the other hand, we may generalize the above solution by putting $P = \phi(m)\,dm$, and integrating from 0 to ∞. Hence, corresponding to the surface stresses

$$p_{zz} = \int_0^\infty J_0(m\varpi)\,\phi(m)\,dm, \quad p_{z\varpi} = 0, \tag{13}$$

we have

$$\left.\begin{aligned}
\theta &= \frac{1}{\lambda+\mu}\int_0^\infty e^{-mz} J_0(m\varpi)\,\phi(m)\,dm \\
w &= -\frac{1}{2\mu}\int_0^\infty z e^{-mz} J_0(m\varpi)\,\phi(m)\,dm \\
&\qquad\qquad - \frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\int_0^\infty e^{-mz} J_0(m\varpi)\,\phi(m)\,\frac{dm}{m} \\
q &= -\frac{1}{2\mu}\int_0^\infty z e^{-mz} J_1(m\varpi)\,\phi(m)\,dm \\
&\qquad\qquad + \frac{1}{2(\lambda+\mu)}\int_0^\infty e^{-mz} J_1(m\varpi)\,\phi(m)\,\frac{dm}{m}
\end{aligned}\right\}. \tag{14}$$

To make (13) represent any arbitrary (symmetrical) distribution of normal traction, we have only to determine $\phi(m)$ suitably. This is effected by means of the formula

$$f(\varpi) = \int_0^\infty J_0(m\varpi)\,m\,dm \int_0^\infty f(\lambda)\,J_0(m\lambda)\,\lambda\,d\lambda, \tag{15}$$

viz., if $f(\varpi)$ be the given surface value of p_{zz}, we must write

$$\phi(m) = m\int_0^\infty f(\lambda)\,J_0(m\lambda)\,\lambda\,d\lambda. \tag{16}$$

Thus, to find the effect of a concentrated normal pressure at the origin, we may suppose that $f(\varpi)$ vanishes for all but infinitesimal values of ϖ, when it becomes infinite in such a manner that

$$\int_0^\infty f(\varpi)\,2\pi\varpi\,d\varpi = -1;$$

this makes
$$\phi(m) = -\frac{m}{2\pi}.$$

The evaluation of the integrals in (14) then follows at once from

the known theorem

$$\int_0^\infty e^{-mz} J_0(m\varpi)\,dm = \frac{1}{\sqrt{(\varpi^2+z^2)}} = \frac{1}{r}, \tag{17}$$

where r denotes distance from the origin. Differentiating this with respect to z and ϖ respectively, we infer

$$\int_0^\infty e^{-mz} J_0(m\varpi)\,m\,dm = \frac{z}{r^3}, \tag{18}$$

$$\int_0^\infty e^{-mz} J_1(m\varpi)\,m\,dm = \frac{\varpi}{r^3}. \tag{19}$$

Again, integrating (19) with respect to z, and determining the additive constant so as to make the result vanish for $z = \infty$, we have

$$\int_0^\infty e^{-mz} J_1(m\varpi)\,dm = \frac{\varpi}{r(r+z)}. \tag{20}$$

Substituting in (14), we find

$$\left.\begin{aligned} \theta &= -\frac{1}{2\pi(\lambda+\mu)}\,\frac{z}{r^3} \\ w &= \frac{1}{4\pi\mu}\,\frac{z^2}{r^3} + \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)}\,\frac{1}{r} \\ q &= \frac{1}{4\pi\mu}\cdot\frac{z\varpi}{r^3} - \frac{1}{4\pi(\lambda+\mu)}\,\frac{\varpi}{r(r+z)} \end{aligned}\right\}, \tag{21}$$

which are the known results for this case.* For the component stresses at any point we deduce

$$\left.\begin{aligned} p_{zz} &= -\frac{3}{2\pi}\,\frac{z^3}{r^5}, \quad p_{z\varpi} = -\frac{3}{2\pi}\,\frac{z^2\varpi}{r^5} \\ p_{\varpi\varpi} &= -\frac{3}{2\pi}\,\frac{z\varpi^2}{r^5} + \frac{\mu}{2\pi(\lambda+\mu)}\,\frac{1}{r(r+z)} \end{aligned}\right\}. \tag{22}$$

These formulæ have been discussed by Boussinesq and others; but one or two simple results relating to the case of incompressibility ($\lambda = \infty$) may be noted. In the first place the differential equation

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* See, for example, Love, *Elasticity*, Vol. I., p. 270.

of the "isostatic" lines in a meridian plane, viz.,

$$\frac{p_{zz}\,dz+p_{z\varpi}\,d\varpi}{dz}=\frac{p_{z\varpi}\,dz+p_{\varpi\varpi}\,d\varpi}{d\varpi}, \tag{23}$$

reduces to
$$(\varpi\,d\varpi+z\,dz)(z\,d\varpi-\varpi\,dz)=0; \tag{24}$$

so that the curves in question consist of concentric circles and radial straight lines.* Again, the values of q and w can be expressed in terms of a displacement-function ψ, viz., we have

$$q=\frac{1}{\varpi}\frac{d\psi}{dz},\quad w=-\frac{1}{\varpi}\frac{d\psi}{d\varpi}, \tag{25}$$

where
$$\psi=\frac{1}{4\pi\mu}\frac{\varpi^2}{r}; \tag{26}$$

the equation of the lines of displacement is therefore

$$\varpi^2=ar.$$

These curves start from the boundary at right angles, and then curve outwards; they have inflexions at distances $\frac{3}{2}a$ from the origin, and ultimately became parabolic. When the substance is not incompressible the lines of displacement are inclined towards the axis at the surface, making an angle $\tan^{-1}(\lambda+2\mu)/\mu$ with it.

Boussinesq has also investigated the case where a perfectly rigid circular cylinder of finite radius (a) presses normally (as *e.g.*, by its own gravity when the surface is horizontal) against the surface. The conditions to be satisfied are then

$$w=\text{const.}\quad\text{for}\quad z=0,\ \varpi<a,$$

and
$$p_{zz}=0\qquad\text{for}\quad z=0,\ \varpi>a.$$

The function $\phi(m)$ which occurs in (13) and (14) must therefore satisfy the conditions

$$\int_0^\infty J_0(m\varpi)\,\phi(m)\,dm=0\qquad[\varpi>a], \tag{27}$$

$$\int_0^\infty J_0(m\varpi)\,\phi(m)\frac{dm}{m}=\text{const.}\quad[\varpi<a]. \tag{28}$$

It is known that

$$\int_0^\infty J_0(m\varpi)\sin ma\,dm=\frac{1}{\sqrt{(a^2-\varpi^2)}},\quad\text{or}\quad 0, \tag{29}$$

---

* Compare the sketch given by Hertz, *Werke*, Vol. I., p. 185.

and $$\int_0^\infty J_0(m\varpi)\frac{\sin ma}{m}\,dm = \tfrac{1}{2}\pi, \quad\text{or}\quad \sin^{-1}\frac{a}{\varpi}, \tag{30}$$

the first or second value being taken, in each case, according as $\varpi \lessgtr a$. The conditions (27), (28) are therefore satisfied by

$$\phi(m) = C\sin ma,$$

which gives an aggregate normal pressure

$$W = -2\pi a C.$$

The surface displacements are accordingly, in terms of W,

$$\left.\begin{aligned} w_0 &= \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)}\frac{W}{a}\int_0^\infty J_0(m\varpi)\frac{\sin ma}{m}\,dm \\ q_0 &= -\frac{1}{4\pi(\lambda+\mu)}\frac{W}{a}\int_0^\infty J_1(m\varpi)\frac{\sin ma}{m}\,dm \end{aligned}\right\}; \tag{31}$$

whence $$\left.\begin{aligned} w_0 &= \frac{\lambda+2\mu}{8\mu(\lambda+\mu)}\frac{W}{a} \\ q_0 &= -\frac{W}{4\pi(\lambda+\mu)a}\frac{a-\sqrt{(a^2-\varpi^2)}}{\varpi} \end{aligned}\right\}\ [\varpi<a], \tag{32}$$

and $$\left.\begin{aligned} w_0 &= \frac{\lambda+2\mu}{8\mu(\lambda+\mu)}\frac{W}{a}\frac{2}{\pi}\sin^{-1}\frac{a}{\varpi} \\ q_0 &= -\frac{W}{4\pi(\lambda+\mu)a}\frac{a}{\varpi} \end{aligned}\right\}\ [\varpi>a].^* \tag{33}$$

In this problem the pressure on the surface of contact increases from the centre to the circumference (where it is infinite) according to the law given by (29). The effect of a pressure distributed *uniformly* over a circular area of radius a is obtained by making $f(\lambda) = -1$ from 0 to a, and $=0$ from a to ∞, in (16). If W be the total pressure, we find for the surface displacements

$$\left.\begin{aligned} w_0 &= \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)}\frac{W}{a}\int_0^\infty J_0(m\varpi)\,J_1(ma)\frac{dm}{m} \\ q_0 &= -\frac{1}{4\pi(\lambda+\mu)}\frac{W}{a}\int_0^\infty J_1(m\varpi)\,J_1(ma)\frac{dm}{m} \end{aligned}\right\}. \tag{34}$$

---

* The evaluation of the second integral in (31) is effected by multiplying both sides of (29) by $\varpi\,d\varpi$ and integrating with respect to ϖ.

The definite integrals can be evaluated in the form of infinite series by means of the formulæ

$$\int_0^\infty e^{-mz} J_0(m\varpi)\, m^n dm = \frac{n!}{r^{n+1}} P_n\left(\frac{z}{r}\right), \tag{35}$$

$$\int_0^\infty e^{-mz} J_1(m\varpi)\, m^n dm = \frac{(n-1)!}{r^{n+2}} P_n'\left(\frac{z}{r}\right)\varpi, \tag{36}$$

where P_n is the symbol of the ordinary zonal harmonic. These identities follow easily from (17).* We thus find

$$w_0 = \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)} \frac{W}{a} F\left(\tfrac{1}{2}, -\tfrac{1}{2}, 1, \frac{\varpi^2}{a^2}\right), \tag{37}$$

for $\varpi < a$, and

$$w_0 = \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)} \frac{W}{\varpi} F\left(\tfrac{1}{2}, \tfrac{1}{2}, 2, \frac{a^2}{\varpi^2}\right), \tag{38}$$

for $\varpi > a$.†

‡ [The foregoing analysis can be adapted to the study of the deformations of an infinite *plate* produced by normal forces applied to its boundaries; the results, however, do not appear to admit of easy reduction.

The most interesting case is that of pure flexure, where the middle surface is unextended. We assume therefore

$$\theta = A \sinh mz\, J_0(m\varpi), \tag{39}$$

the origin being taken in the middle surface. We find

$$\left.\begin{aligned} w &= \left(-\frac{\lambda+\mu}{2\mu} Az\sinh mz + \frac{B}{m}\cosh mz\right) J_0(m\varpi) \\ q &= \left(\frac{\lambda+\mu}{2\mu} Az\cosh mz + \frac{C}{m}\sinh mz\right) J_1(m\varpi) \end{aligned}\right\}, \tag{40}$$

* [They are known results; see Hobson, *Proc. Lond. Math. Soc.*, Vol. xxv., pp. 72, 73.]

† The method of this paper is, of course, not restricted to the case of symmetry about the axis of z; and it would doubtless be possible to work out in a similar manner the effect of a concentrated tangential surface traction. It can hardly be claimed, however, that the method is specially appropriate to this case.

‡ [Added March 15, 1902, in accordance with a suggestion made by Prof. Love at the meeting when the paper was read.]

with the condition
$$B+C=\frac{\lambda+3\mu}{2\mu}A. \tag{41}$$

If h denote the half-thickness, we have, at the surfaces $z=\pm h$,

$$p_{zz}=\pm\tfrac{1}{2}P.J_0(m\varpi),\quad p_{z\varpi}=0, \tag{42}$$

provided

$$\left.\begin{aligned}-\left(\sinh mh+\frac{\lambda+\mu}{\mu}mh\cosh mh\right)A+2B\sinh mh&=\tfrac{1}{2}P\\ \frac{\lambda+\mu}{2\mu}(\cosh mh+2mh\sinh mh)-(B-C)\cosh mh&=0\end{aligned}\right\}. \tag{43}$$

These equations, combined with (41), give A, B, C. In particular, for the deflection w_0 of the middle surface we find

$$w_0=\frac{\frac{\lambda+2\mu}{\lambda+\mu}\cosh mh+\sinh mh}{\sinh 2mh-2mh}\cdot\frac{P}{2m}J_0(m\varpi). \tag{44}$$

This can be generalized as before by writing

$$P=\phi(m)\,dm,$$

and integrating with respect to m between the limits 0 and ∞; and, if the total normal force per unit area (supposed divided equally between the two faces) be denoted by $f(\varpi)$, the value of $\phi(m)$ is as in (16); whence

$$w_0=\tfrac{1}{2}\int_0^\infty\frac{\frac{\lambda+2\mu}{\lambda+\mu}\cosh mh+\sinh mh}{\sinh 2mh-2mh}J_0(m\varpi)\,dm\int_0^\infty f(\lambda)J_0(m\lambda)\lambda d\lambda. \tag{45}$$

As a particular case we may suppose the plate to be horizontal, and to be supported along the circumference of a circle ($r=a$), whilst a load W is applied at the origin. We find

$$w_0=\frac{1}{4\pi}\int_0^\infty\frac{\frac{\lambda+2\mu}{\lambda+\mu}\cosh mh+\sinh mh}{\sinh 2mh-2mh}\{1-J_0(ma)\}J_0(m\varpi)\,dm. \tag{46}]$$

On the Density of Linear Sets of Points. *By* W. H. Young.
Received January 25th, 1902. Communicated February 13th, 1902.

In one of T. Brodén's valuable memoirs on the real functions of a real variable there is a small error, to which it is perhaps worth while to call attention, as the point involved is one possessing some interest in itself. The mistake in question will be found on p. 23 of the memoir entitled "Beiträge zur Theorie der stetigen Funktionen einer reellen Variabeln," *Crelle*, CXVIII. It consists in the tacit assumption that, if each point of a linear set of points is a limiting point on both sides, then the set will be *dense everywhere* (*überall dicht*, *überall condensirt*). A set of points of which every point is a limiting point has been called "dense in itself"; and it is known that the terms "dense in itself" and "everywhere dense" are not simply different terms for characterizing the same type of sets of points.* It will be noted, however, that in the case considered by T. Brodén each point of the set is not merely a limiting point of the set, but a limiting point on both sides. Each point therefore possesses the property that its distance from, *so to speak*, the "next" point on either side is a quantity as small as we please. The conclusion that such a set of points must be distributed over the whole segment of the continuum in which we are operating would seem inevitable to a person unfamiliar with the theory of sets, and even Brodén, who has shown himself a master of subtle points of analysis, including this very class of questions, has fallen inadvertently into this error.

Take, however, any set which is *perfect*† and *nowhere dense*. Omit those of its points which are end points of the intervals of its complementary set of intervals, and we at once get a set of points which is *dense in itself* in Brodén's manner and yet nowhere dense.

This set has the potency‡ of the continuum, whereas the sets of points with which Brodén is concerned are countable. We need, however, merely select from our set a countable set which is every-

* *Cf.* Cantor, *Math. Ann.*, Vol. XXI., p. 575.
† Dense in itself and closed. Cantor, *loc. cit.*
‡ *Mächtigkeit.*

where dense with respect to it, and we shall have a set of points which is at once *countable*, *dense in itself on both sides of every one of its points*, *and yet nowhere dense.*

Suppose, for definiteness, we take the set of numbers obtained by taking all the proper fractions expressed in the binary scale (terminating and non-terminating), and interpreting them in the ternary scale. This set is perfect and nowhere dense,* and those points of it which are limits on one side only are the terminating fractions, and those obtained from them by placing the circulating dot over the final 1. If, therefore, we take all the terminating fractions of the ternary scale, and after expressing them in the binary scale interpret them in the ternary scale, we shall get a countable set contained in our perfect set, and everywhere dense with respect to it. Hence each number of our countable set will be a limit on both sides for numbers of that set, and yet the set will be nowhere dense.

This example is sufficient to prove the existence of sets having the property under discussion, and the general method indicated above is at least theoretically sufficient for the construction of unlimited examples. As, however, it is not at once obvious that we could in this case so arrange the construction that the set ultimately ob-

---

* This is the perfect set of numbers dense nowhere in the segment (0, 1) which is got from H. J. S. Smith's ternary set by derivation (*Proc. Lond. Math. Soc.*, Vol. VI., 1875, p. 148). I take this opportunity of calling attention to two points in the account in Schoenflies' "Bericht ueber die Mengenlehre," p. 101 *et seq.* (*Jahresbericht der deutschen Mathematiker-vereinigung*, Vol. VIII.), which are liable to give a false impression. The first is that H. J. S. Smith's set is not itself perfect, as Schoenflies' introductory remarks would lead one to suppose, the geometrical mode in which it is constructed introducing of necessity isolated points, even if you choose to explicitly include in the set its limiting points, so as to close it, as Volterra subsequently did in a similar example; this Smith does not apparently do; so that his set, as he constructed it is, like his other examples, countable. This oversight renders the account in Schoenflies unnecessarily obscure, commencing as it does with the words "In the interests of history [*sic*] I give here that example of a perfect set of numbers dense nowhere which was constructed by H. J. S. Smith." The interest of the student becomes aroused from more than merely historical motives when he realizes that H. J. S. Smith's ternary *derived* set is to all intents and purposes the same as Cantor's ternary set of numbers introduced by Schoenflies on the following page with the words "the first example of a perfect set dense nowhere which was *consciously* constructed was given by Cantor"; the former set consists in fact of all the ternary fractions involving the figures 0 and 1 only, and the latter set of those involving only 0 and 2. This arithmetical connexion being entirely ignored, this section of the *Bericht* seems wanting in unity of purpose as well as in perspective. Schoenflies' subsequent remarks about the generalization of Cantor's set, when any other base number is adopted, would have their proper place in connexion with Smith's work, who, eight years before Cantor introduced the ideas and the definition of a perfect set, actually adopted a general base $m \geqq 3$, and whose numbers are the m-ary fractions which do not involve the figure $(m-1)$.

tained satisfied Brodén's special requirements,* I propose to give an example built up in Brodén's manner.

Take the segment (0, 1) of the y-axis, and divide it at the point y_1 into two parts, the lower s_{01}, and the upper s_{11}, so that the ratio

$$s_{01} : s_{11} = 1+j_1 : 1-j_1,$$

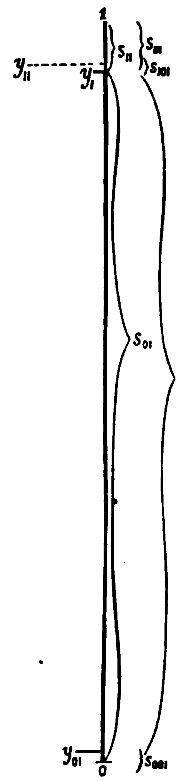
where $$j_1 = 1-\tfrac{1}{8}.$$

Next divide each of the two segments so obtained in precisely the same way, j_2 taking the place of j_1, where

$$j_2 = -\left(1-\frac{1}{8.2^2}\right),$$

and so on, j_n being defined by the equation

$$j_n = (-)^{n-1}\left(1-\frac{1}{8n^2}\right).$$



The new points of division at the end of the second stage we denote by y_{01} and y_{11}, y_{01} lying in s_{01} and y_{11} in s_{11}; the new points at the end of the third stage by y_{001}, y_{011}, y_{101}, y_{111}, and so on. Moreover, the intervals themselves at the end of the second stage will be denoted by s_{001}, s_{011}, s_{101}, s_{111}, and y_{001} will lie in s_{001}, and so for the others. The general law of division and notation is now obvious.† The points of division are called by Brodén *primary points*. Then we assert that the set of primary points is of the type required.

From the method of formation of the s's it is evident that the suffix of the maximum segment at the end of the $(2m-1)$-th stage is $(01)^m$, and at the end of the $2m$-th stage is $(01)^m 1$. Also, whether n be even or odd, the length of the maximum segment at the end of the n-th stage is

$$\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{4^2.2^2}\right)\left(1-\frac{1}{4^2.3^2}\right)\dots\left(1-\frac{1}{4^2.n^2}\right),$$

---

* *Loc. cit.*, p. 22, lines 4, 9.

† The indices of the new points of division introduced at the n-th division are such that, prefixing to each a dot, they are all the binary fractions involving n binary places; the last figure is therefore always a 1; *cf.* Brodén, *loc. cit.*, bottom of p. 22.

which is always greater than $\frac{1}{\sqrt{2}}\frac{4}{\pi}$, but continually approaches this value as n increases. Since each of these maximum segments lies within the preceding one, they form a *sequence*, and determine a definite interval within all of them, free of primary points, and of length $\frac{2\sqrt{2}}{\pi}$. The ends of this interval are, however, never reached by the primary points; they are, in fact, limiting points of the primary set, but not included in it.

Again, starting with any one of the segments left after any number, say n, of stages, we can show in a precisely similar way, by considering the maximum segment in it obtained at each subsequent stage, that it contains within it a definite interval, free of primary points, $\left(\text{whose length is, however, no longer } \frac{2\sqrt{2}}{\pi} \text{ of its own length}\right)$.

Thus we have shown that between every two primary points there is an interval free of primary points, possessing the property that its end points are also not primary points. Moreover, *every primary point is approached on both sides by primary points.*

Hence it follows that any given segment of the segment (0, 1) is either entirely free of primary points or contains an interval entirely free of primary points; so that *the set of primary points is dense nowhere.*

It is evident that the free intervals are the complementary intervals of a perfect set of points having the primary points as a countable set among those points of the perfect set which are limiting points on both sides.

We have purposely taken a definite numerical example, but we might equally well write

$$j_n = (-)^{n-1}\left(1-\frac{2}{p^2n^2}\right),$$

where p is any integer, obtaining in this way a countable set of examples of the type desired, namely, of sets of points nowhere dense and yet consisting entirely of points which are limiting points on both sides and are capable of construction in Brodén's special manner. It will be remarked that our example belongs to the class indicated *a priori* at the commencement of this paper; each set consists of a suitable selection from among the points which are limits on both sides of a certain perfect set nowhere dense. It is easy to see that every example of such a set is theoretically obtainable in this

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Vol. XXXIV.—Nos. 782–786.

Issued August 25, 1902.

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way. For, first, it cannot be closed, as it would then be perfect and nowhere dense, and would therefore involve limiting points on one side only. Next, adding those limiting points of the set not already included, we necessarily get a perfect set nowhere dense, which proves the assertion.

We cannot then obtain the condition that a set of points should be everywhere dense by expressing the fact that the distance of every point from its neighbouring points on either side should be indefinitely small, *unless the set of points obtained is known a priori to be closed.*

In the case of open sets of points, such as those with which Brodén is concerned, constructed by means of binary interpolation in a manner similar to that used in my example, we must express the condition that the length of the maximum segment at the end of the n-th stage in the process of division tends towards the limit zero as n is indefinitely increased.*

Comparing the notation here used with that employed by Brodén (p. 22), we find that

$$j_{n+1} = \frac{v_n - u_n}{v_n + u_n}.$$

The maximum segment at the $(n+1)$-th stage being now evidently

$$\prod_{n=0}^{\infty} \frac{1+|j_{n+1}|}{2},$$

it follows that *the necessary and sufficient condition that the primary set should be dense everywhere is*

$$\prod_{n=0}^{\infty} \frac{1+\left|\dfrac{v_n - u_n}{v_n + u_n}\right|}{2} = 0.$$

This condition must be substituted for those given by Brodén at the bottom of p. 23.†

It will be noticed that Brodén's remarks at the bottom of p. 23 and top of p. 24 down to (46) are still valid, the proper condition

* This is equally easily applied when the binary interpolation is the most general possible.

† $$\prod_0^{\infty}\left(1+\frac{v_n}{u_n}\right) = \infty, \quad \prod_0^{\infty}\left(1+\frac{u_n}{v_n}\right) = \infty.$$

It must be added that this error does not impair the validity of the examples in the subsequent part of the paper.

being still satisfied; (46) must, however, be replaced by the condition that when

$$(45) \quad \lim_{n=\infty} \frac{u_n}{v_n} = 0 \quad \text{or} \quad \lim_{n=\infty} \frac{v_n}{u_n} = 0,$$

the series

$$(46a) \quad \sum_0^\infty \left\{ 1 - \left| \frac{v_n - u_n}{v_n + u_n} \right| \right\}$$

must diverge.

In the example given in this paper

$$\prod_0^{2m-1} \frac{u_n}{s_n} \equiv \prod_1^{2m} \frac{1+j_n}{2} = \frac{1}{4^2 (2m)^2} \prod_1^{2m-1} \frac{1+j_n}{2},$$

and $\displaystyle\prod_0^{2m-2} \frac{u_n}{s_n} \equiv \prod_1^{2m-1} \frac{1+j_n}{2}$

$$= \left(\frac{1}{4^2}\right)^{m-1} \frac{1}{2^2 . 4^2 \dots (2m-2)^2} \left(1 - \frac{1}{4^2}\right) \left(1 - \frac{1}{4^2 . 3^2}\right) \dots \left(1 - \frac{1}{4^2 (2m-1)^2}\right).$$

Hence

$$0 < \prod_0^r \frac{u_n}{s_n} < \left(\frac{1}{4^2}\right)^{r-1}$$

for all values of r after $r = 7$; so that

$$\prod_0^\infty \frac{u_n}{s_n} = 0;$$

similarly

$$\prod_0^\infty \frac{v_n}{s_n} = 0.$$

Thus both Brodén's conditions are satisfied, although, as we saw, the set is nowhere dense. Indeed in our case $\frac{u_n}{v_n}$ is alternately greater and less than any assignable quantity, so that no finite (>0 and $<\infty$) limit of $\frac{u_n}{v_n}$ is possible, while, on the other hand, the condition (46a) is not satisfied, the series being convergent, since

$$\prod_0^\infty \frac{1 + \left| \frac{v_n - u_n}{v_n + u_n} \right|}{2} = \frac{2\sqrt{2}}{\pi}.$$

On Plane Cubics. *By* A. C. DIXON. Received January 29th, 1902. Communicated February 13th, 1902.

This note contains some further developments of the theory of corresponding points on a cubic, as given in Salmon's *Higher Plane Curves*, and of the closely connected theory of three conics.

1. The theory of corresponding points on a plane cubic may very well be introduced by a discussion of the problem to inscribe a complete quadrilateral in the curve, that is to find four straight lines whose six points of intersection lie on the curve.

Let us distinguish the points on the curve by their elliptic arguments, taking a point of inflexion as zero, so that three points u, v, w are collinear if $u+v+w \equiv 0$. The change from this notation to the language of the theory of residuation is quite easy.

Let u, v, w, u', v', w' be the six vertices of a complete quadrilateral, so that $u'vw$, $uv'w$, uvw', $u'v'w'$ are collinear triads of points. Then

$$\begin{aligned} u'+v+w &\equiv 0, \\ u+v'+w &\equiv 0, \\ u+v+w' &\equiv 0, \\ u'+v'+w' &\equiv 0. \end{aligned}$$

Hence $$2u \equiv 2u',$$

and $$u'-u \equiv v'-v \equiv w'-w \equiv \text{a half-period, say } \omega.$$

Thus when one vertex is given the opposite vertex has three possible positions, there being three half-periods, and there are three series of inscribed complete quadrilaterals. Each series is doubly infinite, since two vertices, not opposite, may be arbitrarily chosen. In each series the choice of one vertex, u, determines the opposite vertex, u', uniquely; the relation between these points is reciprocal, and they may be called "corresponding points." Since $2u \equiv 2u'$, the tangents at corresponding points meet on the curve.

By supposing $v = u$, or otherwise, we find that $-u-u'$, the residual point of u, u', corresponds to $-2u$, the point of intersection of the tangents at u, u'.

Now let u_1, v_1, w_1, u_1', v_1', w_1' be the vertices of another quadrilateral of the same system. Then

$$u+v+w \equiv \omega, \quad u_1+v_1+w_1 \equiv \omega,$$
$$u+v+w+u_1+v_1+w_1 \equiv 0,$$

and the six points u, v, w, u_1, v_1, w_1 lie on a conic.

Hence the six lines vw, wu, uv, v_1w_1, w_1u_1, u_1v_1 touch a conic. So do the six lines vw, wu, uv, $v_1'w_1'$, $w_1'u_1$, u_1v_1' in the same way. These conics have five tangents in common, and are therefore the same; similarly $u'v'w'$ touches the same conic. Hence the sides of any two quadrilaterals of the same system touch one conic S.

Again, let any tangent to S cut the cubic in u_2', v_2', w_2', and let u_2, v_2, w_2 be the corresponding points. Then u_2, v_2, w_2, u_2', v_2', w_2' are the vertices of a complete quadrilateral of the system, and the eight lines $u'vw$, $uv'w$, uvw', $u'v'w'$, $u_2'v_2w_2$, $u_2v_2'w_2$, $u_2v_2w_2'$, $u_2'v_2'w_2'$ touch one conic, which must again be S. Thus we have a singly infinite series of quadrilaterals of the system, all circumscribed to S. The same argument would prove the existence of such a series circumscribed to any conic touching the four lines $u'vw$, $uv'w$, uvw', $u'v'w'$ or touching the four sides of any other quadrilateral of the system. Hence the conic S is one of an infinite system, and it is uniquely defined when it is made to touch two arbitrary straight lines, since each of these may be taken as the side of a quadrilateral of the system, so that thus eight tangents are given. The tangential equation to S is therefore of the form $\lambda U+\mu V+\nu W = 0$, U, V, W being definite expressions and λ, μ, ν arbitrary coefficients.

If two of the quadrilaterals circumscribed to S have a pair of opposite vertices in common, the conic S must reduce to this pair of points. The cubic is therefore the locus of the pairs of points included in the system of conics $\lambda U+\mu V+\nu W = 0$, and the two points of any pair are corresponding points on the cubic.

Conversely, if any three conics are given, the tangential equations being $U = 0$, $V = 0$, $W = 0$, the locus of the pairs of points included in the system $\lambda U+\mu V+\nu W = 0$ is a cubic. For, if any straight line α be taken, the conics of the system which touch it touch also three other lines β, γ, δ, and the only points in which α can meet the locus are its intersections with β, γ, δ. The four lines α, β, γ, δ form an inscribed complete quadrilateral, as in the above theory.

Reciprocally, if $U = 0$, $V = 0$, $W = 0$ are ordinary equations, the envelope of the pairs of straight lines included in the system $\lambda U+\mu V+\nu W = 0$ is a curve of the third class.

2. It is known also that the intersections of these pairs of lines lie on a cubic, the Jacobian of the three conics, and the points of this cubic may be taken as "conjugate" in pairs in such a way that the polars with respect to $U = 0$, $V = 0$, $W = 0$ of either of the conjugates meet in the other.

Let A', B', C' be any three collinear points of the Jacobian; A, B, C their respective conjugates; and take $U = 0$, $V = 0$, $W = 0$ to be the three pairs of lines which cut in A', B', C' respectively. Then A lies on the polar of A' with respect to V, that is on the line through B' harmonically conjugate to $B'A'$ with respect to the lines V. C lies on the polar of C' with respect to V. Hence A, C, B' are collinear; so in like manner are B, C, A' and A, B, C', and the six points thus form a complete quadrilateral inscribed in the Jacobian, so that A, A'; B, B'; C, C' are pairs of corresponding points belonging to one system.

The four lines $V = 0$, $W = 0$ form a quadrilateral whose third diagonal is $B'C'$. From the above harmonic relations it follows that the other two diagonals meet in A. These form a conic of the system which we may without loss of generality take to be $V - W = 0$, since it passes through the intersections of $V = 0$, $W = 0$. In like manner we may take $W - U = 0$ to be a pair of straight lines; these will meet in B, and, if k, k' are so chosen that

$$U + kV = 0, \quad V - W + k'(W - U) = 0$$

break up into pairs of lines, the lines of each pair will meet in C. Hence

$$k'(U + kV) + \{V - W + k'(W - U)\} = 0$$

or

$$V(1 + kk') - W(1 - k') = 0$$

will also break up into a pair of lines meeting in C. But this is impossible in general, unless C coincides with B', C' or A, which is not true, and therefore the last equation must be an identity. Thus $k' = 1$, $k = -1$ and $U - V = 0$ represents a pair of lines meeting in C.

The whole figure may be somewhat simplified by projecting $A'B'C'$ to infinity. Then $U = 0$, $V = 0$, $W = 0$ are pairs of parallel lines, say

$$Q_4 Q_1 R_1 R_4 \quad \text{and} \quad Q_3 Q_2 R_3 R_2,$$
$$R_4 R_2 P_2 P_4 \quad \text{and} \quad R_1 R_3 P_1 P_3,$$
$$P_4 P_3 Q_3 Q_4 \quad \text{and} \quad P_2 P_1 Q_2 Q_1.$$

[redacted]

The lines $V-W=0$, $W-U=0$, $U-V=0$ are the pairs of diagonals of the parallelograms $P_1P_2P_4P_3$, $Q_1Q_2Q_3Q_4$, $R_1R_3R_2R_4$, and the following sets of lines are concurrent:—

$$P_1P_4,\quad Q_2Q_4,\quad R_3R_4;$$
$$P_1P_4,\quad Q_1Q_3,\quad R_1R_2;$$
$$P_2P_3,\quad Q_2Q_4,\quad R_1R_2;$$
$$P_2P_3,\quad Q_1Q_3,\quad R_3R_4.$$

Now let the conic $\lambda U+\mu V+\nu W=0$ be "associated" with the point whose coordinates are (λ, μ, ν). Take ABC as triangle of reference, and let the system of coordinates be so chosen that the equation $x+y+z=0$ represents $A'B'C'$. Then the conics associated with A, B, C, A', B', C' are pairs of lines meeting in A', B', C', A, B, C respectively.

Since the lines U form a harmonic pencil with $x+y+z=0$ and $x=0$, we may put

$$U\equiv p\,(x+y+z)^2-qx^2;$$

similarly, we have
$$V\equiv p\,(x+y+z)^2-ry^2;$$
$$W\equiv p\,(x+y+z)^2-sz^2;$$

the coefficients p are all the same since $V-W$, $W-U$, $U-V$ break up into lines.

Then the conic associated with any point in the plane is the polar conic of that point with respect to the cubic

$$p\,(x+y+z)^3-qx^3-ry^3-sz^3=0,$$

which is the locus of a point that lies on its associated conic.

This cubic might have been taken as the foundation of the whole construction, and, if A, B' are made to approach and ultimately coincide, we have the figure discussed in Salmon's *Higher Plane Curves*, p. 153.

It may be noticed that the polar conic of $A'B'C'$ with respect to this last cubic must pass through A, B, C, since it is the locus of points whose polar conics touch $A'B'C'$. The polar lines of A', B', C' touch the Hessian at A, B, C, and, since they are by definition tangents to the polar conic of $A'B'C'$, it follows that this conic touches the Hessian at A, B, C.

The twelve lines $U=0$, $V=0$, $W=0$, $V-W=0$, $W-U=0$, $U-V=0$ all touch the Cayleyan. A rule is given (Salmon, Art. 181) for finding the points of contact; they may be found from our figure

directly as follows. With the notation that was used above, the points Q_1, Q_2, Q_3, Q_4 are common to the polar conics of all points on AC, and R_1, R_2, R_3, R_4 to those of all points on AB. Let any line through any point X meet AC in Y and AB in Z. Then the polar conics of X, Y, Z have four common points, and therefore meet any line, say $Q_3 Q_2 R_3 R_2$, in pairs of points in involution; that is, the polar conic of X meets $Q_3 Q_2 R_3 R_2$ in points in involution with the pairs Q_2, Q_3 and R_2, R_3. Here X may be any point; let it be taken on the Hessian near to A. Then its polar conic consists of two lines, one consecutive to $Q_2 R_2$, and the other to $Q_1 R_1$. In the limit these meet $Q_2 R_2$ in its point of contact with the Cayleyan and in A'. Hence the point of contact is conjugate to A' in the involution to which the pairs Q_2, Q_3 and R_2, R_3 belong. The foci of this involution are points on the Hessian, and so this agrees with the construction given in Salmon.

3. In the above work we have met with three doubly infinite series of conics related to the cubic, namely, the polar conics of lines, all of which have triple contact with the Hessian, the polar conics of points, and the conics which are inscribed in complete quadrilaterals inscribed in the curve. The general equation of a conic of the last series may be found as follows. Let the given cubic be the Hessian of

$$x^3+y^3+z^3+6mxyz = 0;$$

this is to be taken so that the relations between two corresponding points (x_1, y_1, z_1), (x_2, y_2, z_2) of our system are

$$x_1 x_2 + m\,(y_1 z_2 + y_2 z_1) = 0,$$

$$y_1 y_2 + m\,(z_1 x_2 + z_2 x_1) = 0,$$

$$z_1 z_2 + m\,(x_1 y_2 + x_2 y_1) = 0.$$

The tangential equation to these two points is

$$(\xi x_1 + \eta y_1 + \zeta z_1)(\xi x_2 + \eta y_2 + \zeta z_2) = 0,$$

(ξ, η, ζ) being written for current tangential coordinates. This equation may be written

$$x_1 x_2\,(m\xi^2 - \eta\zeta) + y_1 y_2\,(m\eta^2 - \zeta\xi) + z_1 z_2\,(m\zeta^2 - \xi\eta) = 0$$

in virtue of the above conditions, and thus for any conic of the system the equation is

$$\lambda\,(m\xi^2 - \eta\zeta) + \mu\,(m\eta^2 - \zeta\xi) + \nu\,(m\zeta^2 - \xi\eta) = 0,$$

since the system includes an infinity of such pairs of corresponding points. The left-hand side of this equation is the first emanant of the contravariant $m(\xi^3+\eta^3+\zeta^3)-3\xi\eta\zeta$, which may be expressed in Salmon's notation as $4SQ-3TP$, after multiplication by $(1+8m^3)^2$.

Elementary Proof of a Theorem for Functions of several Variables.

By H. F. Baker. Received and read February 13th, 1902.

We are in the habit of assuming that, if an ordinary power series in any number of variables does not vanish for zero values of the variables, the inverse of the series can be expanded in a converging series.

In the following note it is proved that the new series has at least the same range of convergence as the original, provided no zero of the original is contained in this range. A trivial particular case is the convergence of the expansion for $(1+x)^{-n}$ when $|x|<1$, n being a positive integer.

Incidentally a *volume*, of ellipsoidal shape, is found about a point, at which a function of several variables does not vanish, within which no zero of the function exists.

It is pointed out in conclusion that the theorem that every equation has a root is a corollary from the general results.

1. Suppose that the ordinary power series

$$f(x, y) = \sum_{h=0,\, k=0} a_{h,k}\, x^h y^k$$

converges for $|x|<R$, $|y|<S$; let $r<R$, $s<S$, and suppose that for $|x|=r$, $|y|=s$ the derived series

$$f_1(X, Y) = \sum_{m=0} \sum_{n=0} \frac{f^{(m,n)}(x, y)}{m!\, n!}(X-x)^m (Y-y)^n$$

all converge uniformly for $|X-x|=D$, $|Y-y|=E$; let the values of all $|f_1(X, Y)|$ for every set of values

$$|x|=r, \quad |y|=s, \quad X=x+De^{i\theta}, \quad Y=y+Ee^{i\phi}$$

[redacted]

be $< \Pi$; then for every $|x| = r$, $|y| = s$, by a known theorem,

$$\left| \frac{f^{(m,n)}(x, y)}{m!\, n!} \right| \leqq \frac{\Pi}{D^m E^n},$$

and herein we may take for $|f^{(m,n)}(x, y)|$ the upper limit of the absolutely greatest values arising for every $|x| = r$, $|y| = s$.

$$\text{Now } f^{(m,n)}(x, y) = \sum_{h=m} \sum_{k=n} a_{h,k} \frac{h!}{(h-m)!} \frac{k!}{(k-n)!} x^{h-m} y^{k-n}$$

converges uniformly for every $|x| = r$, $|y| = s$; thus we can infer, by a known theorem,

$$|a_{h,k}| \frac{h!\, k!}{(h-m)!\, (k-n)!} \leqq \frac{\Pi\, m!\, n!}{D^m E^n r^{h-m} s^{k-n}}.$$

We have, however, if $D_1 < D$, $E_1 < E$, $r_1 < r$, $s_1 < s$,

$$|a_{hk}|\, r_1^h s_1^k \left(1 + \frac{D_1}{r}\right)^h \left(1 + \frac{E_1}{s}\right)^k$$

$$= |a_{h,k}|\, r_1^h s_1^k \sum_{m=0}^{h} \sum_{n=0}^{k} \frac{h!}{m!\,(h-m)!} \frac{k!}{n!\,(k-n)!} \frac{D_1^m E_1^n}{r^m s^n},$$

of which the right side, in consequence of a previously proved inequality, is

$$\leqq \Pi \left(\frac{r_1}{r}\right)^h \left(\frac{s_1}{s}\right)^k \sum_{m=0}^{h} \sum_{n=0}^{k} \left(\frac{D_1}{D}\right)^m \left(\frac{E_1}{E}\right)^n,$$

that is

$$\lessgtr \Pi \left(\frac{r_1}{r}\right)^h \left(\frac{s_1}{s}\right)^k \frac{1-(D_1/D)^{h+1}}{1-D_1/D} \frac{1-(E_1/E)^{k+1}}{1-E_1/E}$$

$$\lessgtr \Pi \left(\frac{r_1}{r}\right)^h \left(\frac{s_1}{s}\right)^k \frac{1}{(1-D_1/D)(1-E_1/E)}.$$

$$\text{Hence } \sum_{h=0} \sum_{k=0} |a_{hk}|\, r_1^h s_1^k \left(1 + \frac{D_1}{r}\right)^h \left(1 + \frac{E_1}{s}\right)^k$$

$$\lessgtr \Pi \left(1 - \frac{D_1}{D}\right)^{-1} \left(1 - \frac{E_1}{E}\right)^{-1} \left(1 - \frac{r_1}{r}\right)^{-1} \left(1 - \frac{s_1}{s}\right)^{-1}.$$

This shows that the series

$$f(x, y) = \sum_{h=0} \sum_{k=0} a_{h,k} x^h y$$

converges for $|x| = r_1\left(1+\frac{D_1}{r}\right), \quad |y| = s_1\left(1+\frac{E_1}{s}\right),$

quantities of which the upper limits are $r+D$ and $s+E$.

Thus, if $f(x, y)$ be an ordinary power series with a given bicircular region of convergence $|x| < R$, $|y| < S$, and if lower limits, D, E, other than zero, can be assigned for the radii of convergence of derived series $f_1(X, Y)$ about every interior point, then the given series $f(x, y)$ converges in fact within a bicircular region $|x| < R+D$, $|y| < S+E$, of which the radii are greater respectively by D, E than those originally known.

This is merely an extension of the proof and theorem* given for one variable by Harkness and Morley, *Theory of Analytic Functions*, 1898, p. 178; it is now clear that the theorem holds for any number of variables. The proof we have given assumes not only that the derived series $|f_1(X, Y)|$ has a finite upper limit Π_1 for all

$$X = x+De^{i\theta}, \quad Y = y+Ee^{i\phi}$$

for every assigned x, y for which $|x| = r$, $|y| = s$, but that these quantities Π_1 arising for all $|x| = r$, $|y| = s$ have a finite upper limit Π.

2. Now let $$f(x, y) = \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} a_{h,k}\, x^h y^k$$

converge for $|x| \overline{\gtrless} r$, $|y| \overline{\gtrless} s$; and suppose $a_{00} \neq 0$. Put

$$a_{h,k} = -a_{00}\frac{b_{h,k}}{r^h s^k}, \quad x = rt, \quad y = su,$$

and so obtain $$f(x, y) = a_{00}\phi(t, u),$$

where $$\phi(t, u) = 1-\sum_{h=1}^{\infty}\sum_{k=1}^{\infty} b_{h,k}\, t^h u^k$$

converges for $|t| \overline{\gtrless} 1$, $|u| \overline{\gtrless} 1$.

Then the expansion, which we desire to examine,

$$(1-b_{10}t-b_{01}u-\Sigma\Sigma b_{h,k}t^h u^k)^{-1} = 1+B_{10}t+B_{01}u+\Sigma\Sigma B_{h,k}t^h u^k,$$

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* For one variable, Pincherle, "Saggio di una introduzione alla teoria delle *funzioni* analitiche secondo i principii del Prof. C. Weierstrass," *Battaglini's Giornale*, Vol. XVIII., 1880, p. 352.

requires

$$B_{10} = b_{10}, \quad B_{01} = b_{01},$$

$$B_{20} = B_{10}b_{10}+b_{20}, \quad B_{11} = B_{10}b_{01}+B_{01}b_{10}+b_{11}, \quad B_{02} = B_{01}b_{01}+b_{02},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$B_{h,k} = \sum_{m=h}^{0} \sum_{n=k}^{0}{}' B_{m,n} b_{h-m,k-n},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

where the summation $\Sigma\Sigma'$, in the last line written, excludes the combination $m = h$, $n = k$.

Suppose we can find real positive quantities $\beta_{h,k}$ such that

(1) $$\beta_{h,k} \mathrel{\overline{\gtrless}} |b_{h,k}|;$$

(2) the series $$1-\sum_{h=1}^{\infty}\sum_{k=1}^{\infty}\beta_{h,k}t^h u^k$$

converges for an assignable region $|t| <$ some definite quantity, $|u| <$ some definite quantity;

(3) we have an equality

$$(1-\Sigma\Sigma\beta_{h,k}t^h u^k)^{-1} = 1+\sum_{h=1}^{\infty}\sum_{k=1}^{\infty} H_{h,k}t^h u^k$$

holding for an assignable region $|t| <$ some definite quantity, $|u| <$ some definite quantity.

Then we have equations corresponding to those before written, namely,

$$H_{10} = \beta_{10}, \quad H_{01} = \beta_{01},$$

$$H_{20} = H_{10}\beta_{10}+\beta_{20}, \quad H_{11} = H_{10}\beta_{01}+H_{01}\beta_{10}+\beta_{11}, \quad H_{02} = H_{01}\beta_{01}+\beta_{02},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$H_{h,k} = \sum_{m=h}^{0} \sum_{n=k}^{0}{}' H_{m,n}\beta_{h-m,k-n},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

which show that the coefficients $H_{h,k}$ are necessarily real and positive, and

$$H_{10} = \beta_{10} \mathrel{\overline{\gtrless}} |b_{10}| \mathrel{\overline{\gtrless}} |B_{10}|, \quad H_{01} = \beta_{01} \mathrel{\overline{\gtrless}} |b_{01}| \mathrel{\overline{\gtrless}} |B_{01}|,$$

$$H_{20} = H_{10}\beta_{10}+\beta_{20} \mathrel{\overline{\gtrless}} |B_{10}b_{10}|+|b_{20}| \mathrel{\overline{\gtrless}} |B_{20}|,$$

and so on; thus in general

$$H_{h,k} \mathrel{\overline{\gtrless}} |B_{h,k}|,$$

of which we give a formal proof by induction; namely, in consequence of

$$| b_{h-m,\,k-n} | \overline{\overline{<}} \beta_{h-m,\,k-n},$$

and assuming

$$| B_{m,\,n} | \leq H_{m,\,n}$$

for every m from 0 to h, and every n from 0 to k, except only the one combination $m = h$, $n = k$, it follows from

$$| B_{h,\,k} | = \left| \sum_{m=0}^{h} \sum_{n=0}^{k}{}' B_{m,\,n} b_{h-m,\,k-n} \right|$$

$$\overline{\overline{<}} \sum_{m=0}^{h} \sum_{n=0}^{k}{}' | B_{m,\,n} | \, | b_{h-m,\,k-n} |$$

that

$$| B_{h,\,k} | \overline{\overline{<}} \sum_{m=0}^{h} \sum_{n=0}^{k}{}' H_{m,\,n} \beta_{h-m,\,k-n} \overline{\overline{<}} H_{h,\,k},$$

which establishes $| B_{m,\,n} | \overline{\overline{<}} H_{m,\,n}$ for the exceptional case $m = h$, $n = k$; and therefore the general truth of the specified inequality.

It follows therefore from the assumed existence of quantities $\beta_{h,\,k}$ satisfying the three conditions (1), (2), (3) that the series

$$1 + B_{10} t + B_{01} u + \Sigma\Sigma B_{h,\,k} t^h u^k$$

converges within the limits of convergence of the series

$$1 + H_{10} t + H_{01} u + \Sigma\Sigma H_{h,\,k} t^h u^k.$$

Such a set $\beta_{h,\,k}$ can, however, be found as follows. Let M be a positive real quantity greater than (or equal to) the modulus of $\phi(t, u)$ for $|t| = 1$, $|u| = 1$; and take

$$1 - \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \beta_{h,\,k} t^h u^k = 1 - M(t + u + t^2 + tu + u^2 + \ldots),$$

namely, every $\beta_{h,\,k} = M$. Then it follows from a known theorem that

$$| b_{h,\,k} | \leq M \leq \beta_{h,\,k}.$$

And $\left[1 - M(t + u + t^2 + tu + u^2 + \ldots)\right]^{-1}$

$$= \left[1 - M\left(\frac{1}{(1-t)(1-u)} - 1\right)\right]^{-1}$$

$$= \frac{1}{1+M} \frac{1}{1 - \frac{M}{1+M}(1-t)^{-1}(1-u)^{-1}}$$

$$= \frac{1}{1+M} \left\{ 1 + \sum_{\lambda=1}^{\infty} \left(\frac{M}{1+M}\right)^{\lambda} (1-t)^{-\lambda} (1-u)^{-\lambda} \right\};$$

now, when $|t| = \rho < 1$, $|u| = \sigma < 1$, the sum of the moduli of the terms of the expansion of $(1-t)^{-\lambda}(1-u)^{-\lambda}$ is convergent, being equal to $(1-\rho)^{-\lambda}(1-\sigma)^{-\lambda}$; and the sum of the moduli of the terms of the series just written down is

$$\frac{1}{1+M-M(1-\rho)^{-1}(1-\sigma)^{-1}},$$

provided $$(1-\rho)(1-\sigma) > \frac{M}{1+M}.$$

Under this condition, with $|t| = \rho < 1$, $|u| = \sigma < 1$, it is therefore legitimate to arrange the expansion of

$$\left[1-M(t+u+t^2+tu+u^2+\ldots)\right]^{-1}$$

as a power series in t and u. Thus we can conclude that for

$$|x| < r, \quad |y| < s, \quad \left(1-\left|\frac{x}{r}\right|\right)\left(1-\left|\frac{y}{s}\right|\right) > \frac{M}{1+M},$$

we have

$$\frac{1}{f(x, y)} = \frac{1}{a_{00}}\left(1+\sum_{h=1}^{\infty}\sum_{k=1}^{\infty} B_{h,k}t^h u^k\right) = \frac{1}{a_{00}}\left[1+\sum_{h=1}^{\infty}\sum_{k=1}^{\infty} B_{h,k}\left(\frac{x}{r}\right)^h\left(\frac{y}{s}\right)^k\right],$$

where M is not less than the modulus of $f(x, y)/a_{00}$ for $|x| = r$, $|y| = s$.

Let $K >$ modulus of $f(x, y)$ for $|x| \leqq r$, $|y| \leqq s$; we may then take $M = K/|a_{00}|$, and have therefore the expansion which we shall write

$$\frac{1}{f(x, y)} = \frac{1}{a_{00}} + \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} C_{h,k}x^h y^k \tag{A}$$

for $$|x| < r, \quad |y| < s, \quad \left(1-\left|\frac{x}{r}\right|\right)\left(1-\left|\frac{y}{s}\right|\right) > \frac{K}{|a_{00}|+K}.$$

The conditions

$$\rho < 1, \quad \sigma < 1, \quad (1-\rho)(1-\sigma) > \frac{M}{M+1} \tag{1}$$

require $1-\rho > \frac{M}{M+1}$, and therefore $\rho < \frac{1}{M+1}$, and therefore also $\sigma < \frac{1}{M+1}$; putting

$$(1-\rho)\sqrt{\frac{M+1}{M}} = \lambda_0, \quad (1-\sigma)\sqrt{\frac{M+1}{M}} = \mu_0,$$

or $$\rho = 1-\lambda_0\sqrt{\frac{M}{M+1}}, \quad \sigma = 1-\mu_0\sqrt{\frac{M}{M+1}}, \tag{2}$$

we have

$$\sqrt{\frac{M}{M+1}} < \lambda_0 < \sqrt{\frac{M+1}{M}}, \quad \sqrt{\frac{M}{M+1}} < \mu_0 < \sqrt{\frac{M+1}{M}}, \quad \lambda_0\mu_0 > 1. \quad (3)$$

Conversely, if λ_0, μ_0 be any two real positive quantities satisfying (3), the conditions (1) are satisfied in the most general way by values ρ, σ given by (2); which involve in particular $\rho < \frac{1}{M+1}$, $\sigma < \frac{1}{M+1}$. And the series above given holds for

$$|x| \overline{\overline{<}}\, r\left[1-\lambda_0\sqrt{\frac{K}{K+|a_{00}|}}\right], \quad |y| \overline{\overline{<}}\, s\left[1-\mu_0\sqrt{\frac{K}{K+|a_{00}|}}\right].$$

3. It is an incidental consequence that for $|x| \leqq r\rho$, $|y| \leqq s\sigma$, where ρ, σ are determined by (1), or by (2) and (3), there is no pair of values x, y for which $f(x, y)$ is zero; a result which may be proved directly. Assume now that for all $|x| \leqq r$, $|y| \leqq s$, the modulus of $f(x, y)$ remains greater than a definite real positive quantity P; we proceed to show that then the region of convergence of the series

$$\frac{1}{a_{00}} + \Sigma\Sigma\, C_{h,k}\, x^h y^k \qquad \text{(A)}$$

extends, in fact, to within unassignable nearness of $|x| = r$, $|y| = s$.

Take x_1, y_1 so that $|x_1| = r_1 < r$, $|y_1| = s_1 < s$ are circles not lying without the circles of convergence of the last written series (A); then for $|x-x_1| < r-r_1$, $|y-y_1| < s-s_1$ we can write

$$f(x, y) = f(x_1, y_1) + \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \frac{f^{(h,k)}(x_1, y_1)}{h!\,k!}(x-x_1)^h (y-y_1)^k.$$

Applying to this the result previously obtained, it follows, $f(x_1, y_1)$ not being zero, that for

$$|x-x_1| < r-r_1, \quad |y-y_1| < s-s_1,$$

and
$$\left(1-\frac{|x-x_1|}{r-r_1}\right)\left(1-\frac{|y-y_1|}{s-s_1}\right) > \frac{K}{K+|f(x_1, y_1)|}$$

we have
$$\frac{1}{f(x, y)} = \frac{1}{f(x_1, y_1)} + \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} D_{h,k}\,(x-x_1)^h (y-y_1)^k. \qquad \text{(B)}$$

This will therefore, *a fortiori*, be true for

$$|x-x_1| < r-r_1, \quad |y-y_1| < s-s_1,$$

$$\left(1-\frac{|x-x_1|}{r-r_1}\right)\left(1-\frac{|y-y_1|}{s-s_1}\right) > \frac{K}{K+P},$$

for the quantity on the right of the last inequality is greater than $K/\left[K+|f(x_1, y_1)|\right]$, and therefore, as remarked above, true for

$$|x-x_1| \leqq (r-r_1)\left[1-\lambda\sqrt{\frac{K}{K+P}}\right],$$

$$|y-y_1| \leqq (s-s_1)\left[1-\mu\sqrt{\frac{K}{K+P}}\right],$$

where λ, μ are quantities which we may take fixed satisfying only the inequalities

$$\sqrt{\frac{K}{K+P}} < \lambda < \sqrt{\frac{K+P}{K}}, \quad \sqrt{\frac{K}{K+P}} < \mu < \sqrt{\frac{K+P}{K}}, \quad \lambda\mu > 1.$$

The series (B), being equal to the series (A) previously obtained for all points x, y lying within the region of convergence of (B) for which $|x| < r_1$, $|y| < s_1$, is necessarily the derived series of (A).

It follows therefore from an extension theorem proved here (§ 1) that the region of convergence of the series (A) extends beyond the limits $|x| < r_1$, $|y| < s_1$, and is at least as great as given by

$$|x| < r_1+(r-r_1)\left[1-\lambda\sqrt{\frac{K}{K+P}}\right],$$

$$|y| < s_1+(s-s_1)\left[1-\mu\sqrt{\frac{K}{K+P}}\right].$$

Denoting these by $|x| < r_2$, $|y| < s_2$, and putting

$$e_1 = 1-\lambda\sqrt{\frac{K}{K+P}}, \quad e_2 = 1-\mu\sqrt{\frac{K}{K+P}},$$

we may similarly prove that the region of convergence is at least as great as given by

$$|x| < r_2+(r-r_2)\,e_1, \quad |y| < s_2+(s-s_2)\,e_2,$$

and so on. And as at each step we extend the radius of convergence associated with x by the same fraction, e_1, of its deficiency from r, and the radius of convergence associated with y by the same fraction, e_2, of its deficiency from s, it follows that the radii can be extended to be within any assignable nearness respectively of r and s.

We have thus the theorem: If

$$f(x, y) = a_{00} + \Sigma\Sigma a_{hk} x^h y^k$$

be convergent for $|x| \leqq r$, $|y| \leqq s$, and for this range be in absolute value everywhere $<$ the real positive K, and if λ_0, μ_0 be any real positive quantities such that, when

$$A = |a_{00}| = |f(0, 0)|,$$

$$\sqrt{\frac{K}{K+A}} < \lambda_0 < \sqrt{\frac{K+A}{K}}, \quad \sqrt{\frac{K}{K+A}} < \mu_0 < \sqrt{\frac{K+A}{K}}, \quad \lambda_0\mu_0 > 1,$$

then the function $f(x, y)$ does not vanish for any values x, y within the range

$$|x| \leqq r\left[1-\lambda_0\sqrt{\frac{K}{K+A}}\right], \quad |y| \leqq s\left[1-\mu_0\sqrt{\frac{K}{K+A}}\right],$$

and for this range we have a convergent expansion of the form

$$\frac{1}{f(x, y)} = \frac{1}{a_{00}} + \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} C_{h,k} x^h y^k.$$

If further r, s be such that for all values $|x| \leqq r$, $|y| \leqq s$ the function $|f(x, y)|$ is assignedly greater than zero, this expansion is convergent and valid for $|x| < r$, $|y| < s$.

When we apply the same reasoning to the function of n variables

$$f(x_1, \ldots, x_n) = a_{0\ldots0} + \Sigma \ldots \Sigma a_{h_1, h_2, \ldots, h_n} x_1^{h_1}, \ldots, x_n^{h_n},$$

assumed to converge for $|x| \leqq r_1, \ldots, |x_n| \leqq r_n$, and such that for these values $|f(x_1, \ldots, x_n)| < K$, we can prove directly from the theorem

$$|a_{h_1, \ldots, h_n}| \leqq K/r_1^{h_1}, \ldots, r_n^{h_n},$$

or verify indirectly as in the preceding, that there is no set of values $x_1, \ldots, x_n$ for which $f(x_1, \ldots, x_n)$ vanishes lying within the range

$$|x_1| < r_1, \ldots, |x_n| < r_n, \quad \left(1-\frac{|x_1|}{r_1}\right), \ldots, \left(1-\frac{|x_n|}{r_n}\right) > \frac{K}{K+A},$$

where $A = |a_{0\ldots0}|$.

Let $\left(\dfrac{K}{K+A}\right)^{1/n}$ be denoted by ω, and put

$$1-\frac{|x_1|}{r_1} = \lambda_1\omega, \quad \ldots, \quad 1-\frac{|x_n|}{r_n} = \lambda_n\omega;$$

then from $1-\frac{|x_1|}{r_1} < 1$ follows that $\lambda_1 < \omega^{-1}$, and from $1-\frac{|x_1|}{r_1} > \omega^n$ follows $\lambda_1 > \omega^{n-1}$, and so for the others; while clearly $\lambda_1 \ldots \lambda_n > 1$.

Conversely, if $\lambda_1, \ldots, \lambda_n$ be any real quantities such that, for $\omega^n = K/(K+A)$,

$$\omega^{n-1} < \lambda_1 < \omega^{-1}, \ \ldots, \ \omega^{n-1} < \lambda_n < \omega^{-1}, \ \ \lambda_1\lambda_2 \ldots \lambda_n > 1,$$

any set of values such that

$$|x_1| \leqq r_1(1-\lambda_1\omega), \ \ldots, \ |x_n| \leqq r_n(1-\lambda_n\omega)$$

is such that

$$|x_1| < r_1, \ \ldots, \ |x_n| < r_n, \ \ \left(1-\frac{|x_1|}{r_1}\right), \ \ldots, \ \left(1-\frac{|x_n|}{r_n}\right) > \frac{K}{K+A};$$

and there is no set of values in this range for which $f(x_1, \ldots, x_n)$ vanishes. And if R^2 be the least possible value of $\xi_1^2+\ldots+\xi_n^2$ subject to $0 \leqq \xi_1 < 1, \ldots, 0 \leqq \xi_n < 1$, and $(1-\xi_1), \ldots, (1-\xi_n) = \omega^n$, which has a different form according to the magnitude of ω, there is no point for which $f(x_1, \ldots, x_n)$ vanishes in the region given by

$$\frac{|x_1|^2}{r_1^2}+\ldots+\frac{|x_n|^2}{r_n^2} < R^2.$$

For instance, when $n = 2$, if $u = \xi^2+\eta^2$ and $\sigma = \xi+\eta$,

$$\omega^2 = (1-\xi)(1-\eta) = 1-\sigma+\tfrac{1}{2}(\sigma^2-u) = \frac{1-u}{2}+\frac{(1-\sigma)^2}{2}$$

or
$$u = 1-2\omega^2+(1-\sigma)^2.$$

Thus, if upon $(1-\xi)(1-\eta) = \omega^2$ there be points for which $\xi+\eta = 1$, namely, if the equation

$$\xi^2-\xi+\omega^2 = 0$$

have the real roots $\quad \xi = \tfrac{1}{2} \pm \tfrac{1}{2}(1-4\omega^2)^{\frac{1}{2}}$,

that is, if $\omega \leqq \tfrac{1}{2}$, then $u = \xi^2+\eta^2$ has $1-2\omega^2$ for its least value.

If, however, $\omega > \tfrac{1}{2}$, then $(1-\sigma)^2$ is least when $\xi = \eta = 1-\omega$, and then $\xi^2+\eta^2$ has $2(1-\omega)^2$ for its least value.

Thus there are no points for which $f(x_1, x_2)$ vanishes within

$$\frac{|x_1|^2}{r_1^2}+\frac{|x_2|^2}{r_2^2} = R^2,$$

where $R^2 = 1-2\omega^2$ when $\omega \leqq \tfrac{1}{2}$ and $R^2 = 2(1-\omega)^2$ when $\omega > \tfrac{1}{2}$.

As $\omega^2 = K/(K+P)$, and K is any real positive quantity $> |f(x, y)|$ for $|x| \leqq r$, $|y| \leqq s$, we can choose ω as nearly unity as we please, and so always take $R^2 = 2(1-\omega)^2$; but this gives a less extended region than $R^2 = 1-2\omega^2$ when the latter exists. For the greatest value of $2(1-\omega)^2$ when $\omega > \frac{1}{2}$ is $< \frac{1}{2}$, while the least value of $1-2\omega^2$ when $\omega \leqq \frac{1}{2}$ is $\frac{1}{2}$.

4. A very simple example of what precedes is a proof, which may be remarked though it is not new, that the equation

$$F(x) = a_0 + a_1 x + \ldots + a_n x^n = 0$$

is satisfied by a finite value of x. It is supposed that $a_0 \neq 0$.

For, if $F(x) \neq 0$ for $|x| \leqq r$, we can put

$$\psi(x) = \frac{1}{F(x)} = \frac{1}{a_0} + C_1 x + C_2 x^2 + \ldots$$

for $|x| < r$. And then for $|x| = r_1 < r$, if $H \geqq |\psi(x)|$ for $|x| = r_1$, $C_k \leqq H r_1^{-k}$.

Let, then, ϵ be an assigned real positive quantity, and let r be the greatest value of $|x|$ for which $|F(x)| \geqq \epsilon$ for every $|x| \leqq r$; so that, for $|x| \leqq r$,

$$|\psi(x)| = \frac{1}{|F(x)|} \leqq \frac{1}{\epsilon};$$

then

$$|C_k| \leqq \frac{1}{\epsilon r_1^k},$$

wherein r_1 is arbitrarily little less than r.

This shows that, with the given fixed ϵ, r cannot be indefinitely great, since otherwise all the coefficients $C_1, C_2, \ldots$ would be indefinitely small, that is, zero, and $F(x)$ would be a constant.

Note on the Wave Surface of a Dynamical Medium, Æolotropic in all respects. *By* T. J. I'A. Bromwich. Communicated February 13th, 1902. Received, in revised form, March 18th, 1902.

In a paper communicated to the London Mathematical Society (*Proceedings*, Vol. XXXII., 1900, p. 311), Mr. H. M. Macdonald has obtained an expression for the energy-function of a continuous medium transmitting transverse waves; the function being supposed a quadratic function of the three rotations as well as of the six ordinary strains, while the inertia is supposed to be of an isotropic character.*

Shortly after seeing this paper, I met with a note by K. Hensel ("Anwendung der Theorie der Modulsysteme auf ein Problem der Optik," *Crelle's Journal f. d. Math.*, Bd. CVIII., 1891, p. 140) in which Kronecker's idea of systems of moduli was used to simplify Kirchhoff's account of Green's theory of crystalline media. Hensel's treatment suggested an alternative method of attacking Macdonald's problem, which is given in § 2 below. But Hensel's theory depends on a special form of the equations of motion, which was obtained by Kirchhoff; and, naturally, I attempted to construct a corresponding set of equations with the more general energy-function. Kirchhoff's method proved somewhat laborious, and I have simplified it by using the symbolical notation due to Aronhold and Clebsch in their theory of invariants; this is suggested by a paper of Christoffel's (see end of § 1, below).

I have, throughout, supposed the medium to be æolotropic for inertia; and this makes the problem solved by me in § 2 to diverge from Macdonald's (or, rather, from the extension indicated in the closing paragraph of his paper). For I have found the general expression for the energy-function on the hypothesis that two of the waves transmitted in *any* direction shall be purely *rotational;* while Green and Macdonald take the two optical waves to be purely *transversal.* The two hypotheses are the same, so long as the

* At the end of the paper, one result is stated, without proof, for the case of a medium which is æolotropic as regards inertia.

x 2

medium is isotropic for inertia; but they differ in the case of æolotropic inertia. The rotational hypothesis admits of a simple comparison with the transversal, by comparing each of them with the electromagnetic theory; then the elastic displacement on the rotational hypothesis, corresponds to the magnetic (or electric) force; while, if the optical waves are transversal, the analogue of the elastic displacement is the magnetic (or electric) induction.

In § 3 I have obtained the equation to the general wave-surface, on the hypotheses already explained; this equation does not seem to have been found previously, except on the electromagnetic theory, and then only by the aid of vector-analysis.

In conclusion (§ 4), I have shown that the analogous results on the hypothesis of transversality can be obtained from those found in §§ 2, 3; the final expression for the potential energy per unit volume becomes extremely complicated; but the equation to the wave surface is only slightly changed from that found in § 2.

1. *Equations of Wave Motion in a general Crystalline Medium.*

The medium is supposed to be elastic, but æolotropic in all respects, so that, if the component displacements at (x, y, z) are u_1, u_2, u_3, then the kinetic energy per unit volume is T, a general quadratic function of the three velocities

$$\frac{\partial u_1}{\partial t}, \quad \frac{\partial u_2}{\partial t}, \quad \frac{\partial u_3}{\partial t},$$

say

$$2T = a_{11}\left(\frac{\partial u_1}{\partial t}\right)^2 + 2a_{12}\frac{\partial u_1}{\partial t}\frac{\partial u_2}{\partial t} + \dots$$

$$= \Sigma a_{rs}\frac{\partial u_r}{\partial t}\frac{\partial u_s}{\partial t} \quad (r, s = 1, 2, 3).$$

The medium is, however, supposed to be homogeneous, so that the cofficients a_{rs} do not vary from point to point.

Similarly, the potential energy per unit volume is V, a general quadratic function of the six strains (e, f, g, a, b, c) and of the three rotations $(\frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_3)$; the coefficients in V being the same at all points of the medium. We write, as usual,

$$e = \frac{\partial u_1}{\partial x}, \quad a = \frac{\partial u_3}{\partial y} + \frac{\partial u_2}{\partial z}, \quad \omega_1 = \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \quad \text{\&c.}$$

For the present, however, it will be convenient to use (instead of

a, e, ω_1, &c.) the nine first differential coefficients*

$$v_1 = \frac{\partial u_1}{\partial x}, \quad v_2 = \frac{\partial u_1}{\partial y}, \quad v_3 = \frac{\partial u_1}{\partial z},$$

$$v_4 = \frac{\partial u_2}{\partial x}, \quad v_5 = \frac{\partial u_2}{\partial y}, \quad v_6 = \frac{\partial u_2}{\partial z},$$

$$v_7 = \frac{\partial u_3}{\partial x}, \quad v_8 = \frac{\partial u_3}{\partial y}, \quad v_9 = \frac{\partial u_3}{\partial z}.$$

Then the equations of motion, by the principle of least Action, are

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{u}_1}\right) = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial v_1}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial v_2}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial v_3}\right),$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{u}_2}\right) = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial v_4}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial v_5}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial v_6}\right),$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{u}_3}\right) = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial v_7}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial v_8}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial v_9}\right).$$

In order to simplify the manipulation of these equations, let us suppose that*

$$2V = \Sigma c_{rs} v_r v_s \quad (r, s = 1, 2, \ldots, 9),$$

where $$c_{rs} = c_{sr},$$

and let us introduce the Aronhold-Clebsch symbolical notation, writing

$$2V = (\Sigma c_r v_r)^2 = (cv)^2,$$

where now the quantities c_r are to be supposed devoid of meaning until actually multiplied together, and then

$$c_r^2 = c_{rr}, \quad c_r c_s = c_{rs} = c_s c_r \quad (r, s = 1, 2, \ldots, 9).$$

It follows at once that

$$\frac{\partial V}{\partial v_r} = c_r\,(cv) \quad (r = 1, 2, \ldots, 9),$$

and so, since the quantities c_{rs} are *constants*, we have

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial v_1}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial v_2}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial v_3}\right) = \left(c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial y} + c_3\frac{\partial}{\partial z}\right)(cv),$$

with two more similar equations.

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* It will be found useful (in §§ 2, 3) to use v_1, v_2, v_3 in a totally different sense; but this need hardly cause confusion, as the two meanings will not occur in the same piece of work. The same remark applies to the quantities c_{rs}.

Thus the three equations of motion are now

$$a_{11}\frac{\partial^2 u_1}{\partial t^2}+a_{12}\frac{\partial^2 u_2}{\partial t^2}+a_{13}\frac{\partial^2 u_3}{\partial t^2}=\left(c_1\frac{\partial}{\partial x}+c_2\frac{\partial}{\partial y}+c_3\frac{\partial}{\partial z}\right)(cv),$$

$$a_{21}\frac{\partial^2 u_1}{\partial t^2}+a_{22}\frac{\partial^2 u_2}{\partial t^2}+a_{23}\frac{\partial^2 u_3}{\partial t^2}=\left(c_4\frac{\partial}{\partial x}+c_5\frac{\partial}{\partial y}+c_6\frac{\partial}{\partial z}\right)(cv),$$

$$a_{31}\frac{\partial^2 u_1}{\partial t^2}+a_{32}\frac{\partial^2 u_2}{\partial t^2}+a_{33}\frac{\partial^2 u_3}{\partial t^2}=\left(c_7\frac{\partial}{\partial x}+c_8\frac{\partial}{\partial y}+c_9\frac{\partial}{\partial z}\right)(cv).$$

If the medium is transmitting plane waves, in a direction l, m, n, with velocity v, we have

$$\frac{u_1}{a_1}=\frac{u_2}{a_2}=\frac{u_3}{a_3}=f(lx+my+nz-vt),$$

and so we can write

$$v_1=la_1f',\quad v_2=ma_1f',\quad v_3=na_1f',$$

$$v_4=la_2f',\quad v_5=ma_2f',\quad v_6=na_2f',$$

$$v_7=la_3f',\quad v_8=ma_3f',\quad v_9=na_3f'.$$

Thus, if

$$P_1=lc_1+mc_2+nc_3,$$

$$P_2=lc_4+mc_5+nc_6,$$

$$P_3=lc_7+mc_8+nc_9,$$

we have

$$(cv)=(P_1a_1+P_2a_2+P_3a_3)f',$$

and then the equations of motion yield (after division by f') the results

$$v^2(a_{11}a_1+a_{12}a_2+a_{13}a_3)=P_1(P_1a_1+P_2a_2+P_3a_3),$$

$$v^2(a_{21}a_1+a_{22}a_2+a_{23}a_3)=P_2(P_1a_1+P_2a_2+P_3a_3),$$

$$v^2(a_{31}a_1+a_{32}a_2+a_{33}a_3)=P_3(P_1a_1+P_2a_2+P_3a_3).$$

But these are equivalent to the three equations*

$$v^2\frac{\partial A}{\partial a_r}=\frac{\partial B}{\partial a_r}\qquad(r=1,2,3),$$

* It is not difficult to see that, by substituting for u_1, u_2, u_3 in terms of a_1, a_2, a_3, *before* varying the Action, the expression for the Action is (v^2A-B), save for numerical factors. Thus our three equations can also be found by varying a_1, a_2, a_3 and making the Action stationary when expressed in terms of a_1, a_2, a_3. Unfortunately, this process is not, in general, a valid method of finding the equations of motion; and, in fact, it seems that its correctness in our special case is due to the assumptions that V is *quadratic* in the *first* differential coefficients of u_1, u_2, u_3. *For* the sake of rigour, we must adopt the longer process given in the text.

where
$$A = \Sigma a_{r,s} a_r a_s \quad (r, s = 1, 2, 3),$$
$$B = (P_1 a_1 + P_2 a_2 + P_3 a_3)^2.$$

But $(P_1 a_1 + P_2 a_2 + P_3 a_3)$ is obtained from (cv) by substituting la_1, ma_1, na_1, &c., in place of v_1, v_2, v_3, &c.; and thus B is obtained by making the same substitutions in $2V$. Now we supposed that $2V$ was expressed in terms of e, f, g; a, b, c; $\omega_1, \omega_2, \omega_3$, and so, to obtain B, we must make the substitutions

$$\begin{array}{llllll} la_1, & ma_2, & na_3 & \text{for} & e,\ f,\ g, & \text{respectively;} \\ (ma_3+na_2), & (na_1+la_3), & (la_2+ma_1) & ,, & a,\ b,\ c, & ,, \\ (ma_3-na_2), & (na_1-la_3), & (la_2-ma_1) & ,, & \omega_1,\ \omega_2,\ \omega_3, & ,, \end{array}$$

in the original expression for $2V$. It is clear that the coefficients in B are quadratic functions of l, m, n (the direction-cosines of the wave fronts) and *not* constants of the medium.

These equations for v^2 are the generalization of those given by Kirchhoff in his account of Green's theory;* Kirchhoff's equations can be derived from ours by omitting the rotational terms in $2V$ (*i.e.*, those containing $\omega_1, \omega_2, \omega_3$), and taking the medium to be isotropic for inertia, so that we can write

$$A = \rho\,(a_1^2 + a_2^2 + a_3^2).$$

The use of the Aronhold-Clebsch notation in the above work was suggested by Christoffel's paper "Ueber die Fortpflanzung von Stössen durch elastische feste Körper" (*Annali di Matem.*, t. VIII., Ser. 2, 1877, p. 193; see Love's *Elasticity*, Arts. 73–78). But, in other respects, the treatment is not the same as Christoffel's: for his work relates to the propagation of impulses, instead of continuous waves, through the medium.

2. *Restriction of the Optical Waves to be Rotational, and Deduction of a corresponding Expression for the Potential Energy.*

Green, in his original treatment of the problem,† assumed that the optical waves were those in which the displacement was in the wave front; or, in the notation used before,

$$la_1 + ma_2 + na_3 = 0$$

* "Ueber die Reflexion und Brechung des Lichtes an der Grenze krystallinischer Mittel," *Abhandlungen der Berl. Akad.*, 1876, or *Gesammelte Abhandlungen*, p. 352; the work is reproduced in his lectures on "Optics" (*Vorlesungen über math. Physik*, Bd. II., *Optik*, Leipzig, 1893, pp. 192–196).

† *Trans. Camb. Phil. Soc.*, 1839; *Mathematical Papers*, p. 293.

is Green's condition for an optical wave. In the case of a medium, such as Green's, which is isotropic for inertia, this condition is equivalent to the assumption that the two optical waves are rotational waves only (and so the direction of vibration in the third or non-optical wave is normal to the wave fronts). But in a medium which is æolotropic for inertia this equivalence no longer holds good. I shall proceed to investigate the consequences of assuming the optical waves to be *rotational*, and not transverse; thus we may expect to obtain results analogous to those of the electro-magnetic theory when the medium is both electrically and magnetically æolotropic; the analogue of the elastic displacement being the magnetic (or electric) force.

In § 4 I shall show that the results obtained here can be applied to investigate the consequences of Green's hypothesis of transversality when the medium is æolotropic for inertia.

We have seen that the equations of motion may be put in the form

$$v^2\frac{\partial A}{\partial a_1}-\frac{\partial B}{\partial a_1}=0,\quad v^2\frac{\partial A}{\partial a_2}-\frac{\partial B}{\partial a_2}=0,\quad v^2\frac{\partial A}{\partial a_3}-\frac{\partial B}{\partial a_3}=0.$$

Now these equations are of precisely the same type as those which occur in the theory of reducing the two quadratic forms A, B to sums of the same squares; and, in this analogy, the three quantities v_1^2, v_2^2, v_3^2 (which are the squares of the velocities of the three principal waves) will appear as the roots of the determinantal equation of the algebraic problem.

Again, owing to the connexion of A and B with the kinetic and potential energies, it follows that both these must be *definite** quadratic forms; thus, by a theorem of Weierstrass's, the quantities v_1^2, v_2^2, v_3^2 are real and positive.† Further, three real linear combinations of a_1, a_2, a_3 can be found (say β_1, β_2, β_3), such that

$$A=\beta_1^2+\beta_2^2+\beta_3^2,$$

$$B=v_1^2\beta_1^2+v_2^2\beta_2^2+v_3^2\beta_3^2,$$

and it is to be observed that, in virtue of another theorem of

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* That is, A and B can never be negative and can only vanish if a_1, a_2, a_3 are *all* zero.

† *Monatsbericht d. Akad. zu Berlin*, 4 Mar., 1858; *Math. Werke*, Bd. I., p. 233. The theorem quoted occurs in § 3 of the paper.

Weierstrass's, an equality between any two of the quantities v_1^2, v_2^2, v_3^2 will not affect the result.*

Since the reduced equations of motion in terms of a_1, a_2, a_3 merely express the fact that (v^2A-B) is stationary, it follows that the equations of motion in terms of β_1, β_2, β_3 must express the same fact, and are, accordingly,

$$(v^2-v_1^2)\,\beta_1 = 0, \quad (v^2-v_2^2)\,\beta_2 = 0, \quad (v^2-v_3^2)\,\beta_3 = 0.$$

Hence the three possible types of plane waves are given by

$$\begin{aligned} &\text{(i.)} \quad \beta_2 = 0, \quad \beta_3 = 0, \quad v = v_1\,;\\ &\text{(ii.)} \quad \beta_3 = 0, \quad \beta_1 = 0, \quad v = v_2\,;\\ &\text{(iii.)} \quad \beta_1 = 0, \quad \beta_2 = 0, \quad v = v_3. \end{aligned}$$

Now our hypothesis is that two of these three waves shall be purely rotational; take the two to be (i.) and (ii.). It follows that β_1 and β_2 are linear functions of ϖ_1, ϖ_2, ϖ_3 only, where

$$\varpi_1 = ma_3 - na_2, \quad \varpi_2 = na_1 - la_3, \quad \varpi_3 = la_2 - ma_1,$$

so that

$$l\varpi_1 + m\varpi_2 + n\varpi_3 = 0.$$

We have then to find β_3, so that A may take the form $(\beta_1^2+\beta_2^2+\beta_3^2)$, where β_1 and β_2 are of the type just described; a direct solution of this algebraic question is easy,† but seems unnecessary, as we have the familiar identity

$$AA_0 - \gamma^2 = \Sigma c_{rs}\,\varpi_r\,\varpi_s \quad (r,\, s = 1,\, 2,\, 3),$$

where

$$\begin{aligned} A_0 &= a_{11}l^2 + 2a_{12}lm + \ldots,\\ \gamma &= \tfrac{1}{2}\left(l\frac{\partial A}{\partial a_1} + m\frac{\partial A}{\partial a_2} + n\frac{\partial A}{\partial a_3}\right)\\ &= \tfrac{1}{2}\left(a_1\frac{\partial A_0}{\partial l} + a_2\frac{\partial A_0}{\partial m} + a_3\frac{\partial A_0}{\partial n}\right), \end{aligned}$$

and c_{rs} is the minor (with proper sign) of a_{rs} in $|a_{rs}|$, the determinant of the coefficients of A.

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* This theorem occurs in § 4 of the paper quoted, and assumes that at least one of the two forms A, B is *definite*.

† One method is to substitute in A from the equations

$$a_1 = (\varpi_2 + la_3)/n, \quad a_2 = (-\varpi_1 + ma_3)/n,$$

and then collect all the terms in a_3 into one square, by "completing the square." This square is then found to reduce to γ^2/A_0, which proves the uniqueness of the reduction.

It follows that we must have*

$$\beta_1^2+\beta_2^2 = (\Sigma c_{rs}\varpi_r\varpi_s)/A_0, \quad \beta_3 = \gamma A_0^{-\frac{1}{2}},$$

for this arrangement satisfies the rotational hypothesis; and it is easy to show that no other will satisfy the condition. Hence the expression for B must be

$$B = f(\varpi_1, \varpi_2, \varpi_3)+v_3^2\gamma^2/A_0.$$

Further, our hypothesis is to be verified for *all* directions of propagation of the waves; and so this expression for B is to hold for *all* values of the ratios $l:m:n$; and the problem is to deduce the most general expression for V, subject to this condition.

Now B is obtained from $2V$ by substituting $l\alpha_1$ $(=e')$ in place of e, $m\alpha_3+n\alpha_2$ $(=a')$ in place of a, $m\alpha_3-n\alpha_2$ $(=\varpi_1)$ in place of ω_1, and so on. Thus B is a quadratic function of e', f', g', a', b', c', ϖ_1, ϖ_2, ϖ_3 with constant coefficients; and we have to determine its special form in order that it may reduce to the shape given by the last equation, as a consequence of the relations amongst these nine quantities. Accordingly we must find first all the quadratic relations (with constant coefficients) between e', ..., ϖ_3. A straightforward way of doing this is to substitute

$$\alpha_1 = (lg'+n\varpi_2)/n^2, \quad \alpha_2 = (mg'-n\varpi_1)/n^2, \quad \alpha_3 = g'/n,$$

and then, if we write $\lambda = l/n, \quad \mu = m/n,$

for brevity, it will be seen that

$$a' = 2\mu g'-\varpi_1, \quad e' = \lambda^2 g'+\lambda\varpi_2,$$
$$b' = 2\lambda g'+\varpi_2, \quad f' = \mu^2 g'-\mu\varpi_1,$$
$$c' = 2\lambda\mu g'+\mu\varpi_2-\lambda\varpi_1, \quad \varpi_3 = -\lambda\varpi_1-\mu\varpi_2.$$

Eliminating λ, μ, we find that the *only independent relations* (not involving the ratios $l:m:n$) are

$$b'^2-4e'g' = \varpi_2^2, \quad a'^2-4f'g' = \varpi_1^2, \quad 2c'g'-a'b' = \varpi_1\varpi_2,$$
$$b'\varpi_1+a'\varpi_2+2g'\varpi_3 = 0$$

(it being understood that g' is not zero, or these would not be independent).

* Since $\beta_3 = 0$ in each of the rotational waves, it follows that $\gamma = 0$ in each of them. Hence our condition is equivalent to

$$\alpha_1(a_{12}l+a_{12}m+a_{13}n)+\alpha_2(a_{21}l+a_{22}m+{}_{23}n)+\alpha_3(a_{31}l+a_{32}m+a_{33}n) = 0,$$

which is obviously quite distinct from Green's condition

$$l\alpha_1+m\alpha_2+n\alpha_3 = 0.$$

From these four relations we can deduce five others, and no more,* and the five new ones (not, of course, independent of the old ones) just complete a symmetrical set of nine, and are

$$c'^2-4e'f'=\varpi_3^2, \quad 2a'e'-b'c'=\varpi_2\varpi_3, \quad 2b'f'-c'a'=\varpi_3\varpi_1,$$

$$2e'\varpi_1+c'\varpi_2+b'\varpi_3=0, \quad c'\varpi_1+2f'\varpi_2+a'\varpi_3=0.$$

It follows that the most general expression for B (on the rotational hypothesis) is

$$\begin{aligned} B = v_3^2\gamma^2/A_0 &+ L_{11}(a'^2-4f'g')+2L_{12}(2c'g'-a'b')+\dots \\ &+2L_1(2e'\varpi_1+c'\varpi_2+b'\varpi_3)+\dots \\ &+k_{11}\varpi_1^2+2k_{12}\varpi_1\varpi_2+\dots, \end{aligned}$$

where $\gamma = a_{11}e'+a_{22}f'+a_{33}g'+a_{23}a'+a_{31}b'+a_{12}c'.$

Hence the most general expression for V is

$$\begin{aligned} V = \tfrac{1}{2}L\Delta^2 &+ \tfrac{1}{2}L_{11}(a^2-4fg)+L_{12}(2cg-ab)+\dots \\ &+L_1(2e\omega_1+c\omega_2+b\omega_3)+\dots \\ &+\tfrac{1}{2}k_{11}\omega_1^2+k_{12}\omega_1\omega_2+\dots, \end{aligned}$$

where $\Delta = a_{11}e+a_{22}f+a_{33}g+a_{23}a+a_{31}b+a_{12}c,$

and the coefficients of V must be constants (*i.e.*, independent of the quantities l, m, n). Thus $v_3^2 = LA_0$, and so the velocity of the third (or non-optical) wave is not now independent of the direction of propagation, which is the case in a medium which has isotropic properties of inertia.

The expression just found for V reduces to Macdonald's if we assume isotropy of inertia; of course it should do so, for, as stated before, if the medium is isotropic for inertia, then the rotational and transverse hypotheses are identical. This is, indeed, obvious by

* That is, no more *quadratic* relations can be found; one obvious cubic relation is

$$\begin{vmatrix} 2e', & c', & b' \\ c', & 2f', & a' \\ b', & a', & 2g' \end{vmatrix} = 0,$$

which we get by eliminating the ratios $\varpi_1 : \varpi_2 : \varpi_3$ from the three equations analogous to

$$2e'\varpi_1+c'\varpi_2+b'\varpi_3=0.$$

observing that when

$$a_{11} = a_{22} = a_{33}, \quad a_{23} = a_{31} = a_{12} = 0$$

the condition $\gamma = 0$ reduces to the form

$$la_1 + ma_2 + na_3 = 0.$$

3. *Determination of the Wave Surface.*

Taking the value for V just found, we see that the corresponding expression for B is

$$B = L\gamma^2 + \Sigma d_{rs}\varpi_r\varpi_s \quad (r, s = 1, 2, 3),$$

where $$d_{rs} = L_{rs} + k_{rs},$$

and, as before, the expression for A is

$$A = \gamma^2/A_0 + (\Sigma c_{rs}\varpi_r\varpi_s)/A_0.$$

In order to find the velocities of propagation of the two optical waves (*i.e.*, the rotational waves) we make $(v^2A - B)$ stationary for variations of ϖ_1, ϖ_2, ϖ_3, subject to the conditions

$$l\varpi_1 + m\varpi_2 + n\varpi_3 = 0,$$

$$\gamma = 0.$$

Before actually writing down the quadratic which determines the two velocities, it will be convenient to introduce some symbols which we shall find useful in what follows. We write first

$$\Delta = |a_{rs}|,$$

the determinant of the quadratic form A.

Next, let us put $$p_{rs} = c_{rs}/\Delta \quad (r, s = 1, 2, 3).$$

The object of using p_{rs} in place of c_{rs} is that we may have a *reciprocal* relation between a_{rs} and p_{rs}; for now

$$a_{rs} = P_{rs}/|p_{rs}| \quad \text{and} \quad \Delta = |a_{rs}| = 1/|p_{rs}|,$$

where P_{rs} is the minor of p_{rs} in the determinant $|p_{rs}|$. It should be remarked that Δ cannot vanish (because A is a *definite* quadratic form), and so none of the quantities p_{rs} can be infinite.

Suppose next that q_{rs} is a set of quantities derived from d_{rs}, in the same way as p_{rs} is derived from a_{rs}; so that, with an analogous notation,

$$q_{rs} = D_{rs}/|d_{rs}|, \quad d_{rs} = Q_{rs}/|q_{rs}|,$$

$$\Delta' = |d_{rs}| = 1/|q_{rs}|.$$

With these symbols we have, putting $\gamma = 0$,

$$v^2A - B = \Sigma\,(\lambda p_{rs} - d_{rs})\,\varpi_r\varpi_s,$$

where $$\lambda = v^2\Delta/A_0.$$

Now, making $(v^2A - B)$ stationary for variations of $\varpi_1, \varpi_2, \varpi_3$, subject to the condition

$$l\varpi_1 + m\varpi_2 + n\varpi_3 = 0,$$

it will be found that the quadratic for v^2 is

$$\begin{vmatrix} \lambda p_{11} - d_{11}, & \lambda p_{12} - d_{12}, & \lambda p_{13} - d_{13}, & l \\ \lambda p_{21} - d_{21}, & \lambda p_{22} - d_{22}, & \lambda p_{23} - d_{23}, & m \\ \lambda p_{31} - d_{31}, & \lambda p_{32} - d_{32}, & \lambda p_{33} - d_{33}, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0.$$

This determinant will be denoted by $F(\lambda)$, and now the wave surface is to be found as the envelope of the plane

$$lx + my + nz = vt,$$

v being given as a function of l, m, n, by the equation

$$F(\lambda) = 0.$$

Several attempts to determine this envelope directly having failed, I have been obliged to make use of some properties of invariants and covariants in order to obtain the desired result.

Darboux* has proved that we have, identically, for all values of λ,

$$-\frac{F(\lambda)}{|\lambda p_{rs} - d_{rs}|} = \frac{L^2}{\lambda - \lambda_1} + \frac{M^2}{\lambda - \lambda_2} + \frac{N^2}{\lambda - \lambda_3},$$

where $\lambda - \lambda_1, \lambda - \lambda_2, \lambda - \lambda_3$ are the factors of the determinant $|\lambda p_{rs} - d_{rs}|$, and L, M, N are certain linear combinations of l, m, n not containing λ. Taking the special cases $\lambda = \infty$, $\lambda = 0$, it will be found that

$$A_0 = a_{11}l^2 + 2a_{12}lm + \ldots = L^2 + M^2 + N^2$$

and $$q_{11}l^2 + 2q_{12}lm + \ldots = L^2/\lambda_1 + M^2/\lambda_2 + N^2/\lambda_3,$$

where we have used the relations connecting p_{rs} with a_{rs} and d_{rs} with q_{rs}.

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* Liouville's *Journal de Math.*, t. xix. (2me Sér.), 1874 (§ VIII.); a reproduction of this part of Darboux's paper is given in Scott's *Theory of Determinants* (p. 153 of the first edition). It should be observed that $\Sigma p_{rs}\varpi_r\varpi_s$, $\Sigma d_{rs}\varpi_r\varpi_s$ are *definite* quadratic forms; and so an equality amongst $\lambda_1, \lambda_2, \lambda_3$ does not invalidate the theorem. But for this, if λ_2 were equal to λ_1, there might be a term in $1/(\lambda - \lambda_1)^2$ on the right.

Suppose now that X, Y, Z are linear functions of x, y, z, such that

$$lx+my+nz = LX+MY+NZ;$$

then our problem has been reduced to finding the envelope of

$$LX+MY+NZ = vt,$$

where v is given in terms of L, M, N by the equations

$$\frac{L^2}{\lambda-\lambda_1}+\frac{M^2}{\lambda-\lambda_2}+\frac{N^2}{\lambda-\lambda_3} = 0,$$

$$\lambda = v^2\Delta/A_0 = v^2\Delta/(L^2+M^2+N^2).$$

This gives a Fresnel's wave surface

$$\frac{\lambda_1 X^2}{\kappa-\lambda_1}+\frac{\lambda_2 Y^2}{\kappa-\lambda_2}+\frac{\lambda_3 Z^2}{\kappa-\lambda_3} = 0,$$

where $\kappa = (X^2+Y^2+Z^2)\,\Delta/t^2.$

The last equation has now to be expressed in terms of x, y, z. We note that, whatever κ may be, we have

$$(\kappa q_{11}-a_{11})\,l^2+2\,(\kappa q_{12}-a_{12})\,lm+\ldots$$
$$= (\kappa/\lambda_1-1)\,L^2+(\kappa/\lambda_2-1)\,M^2+(\kappa/\lambda_3-1)\,N^2.$$

Hence, as $lx+my+nz = LX+MY+NZ,$

we deduce the identity, for all values of κ,

$$\frac{-1}{|\kappa q_{rs}-a_{rs}|}\begin{vmatrix} \kappa q_{11}-a_{11}, & \kappa q_{12}-a_{12}, & \kappa q_{13}-a_{13}, & x \\ \kappa q_{21}-a_{21}, & \kappa q_{22}-a_{22}, & \kappa q_{23}-a_{23}, & y \\ \kappa q_{31}-a_{31}, & \kappa q_{32}-a_{32}, & \kappa q_{33}-a_{33}, & z \\ x, & y, & z, & 0 \end{vmatrix}$$
$$= \frac{X^2}{\kappa/\lambda_1-1}+\frac{Y^2}{\kappa/\lambda_2-1}+\frac{Z^2}{\kappa/\lambda_3-1}.$$

Putting $\kappa = 0$, this gives

$$p_{11}x^2+2p_{12}xy+\ldots = X^2+Y^2+Z^2.$$

Thus the equation to the wave surface is

$$\begin{vmatrix} \kappa q_{11}-a_{11}, & \kappa q_{12}-a_{12}, & \kappa q_{13}-a_{13}, & x \\ \kappa q_{21}-a_{21}, & \kappa q_{22}-a_{22}, & \kappa q_{23}-a_{23}, & y \\ \kappa q_{31}-a_{31}, & \kappa q_{32}-a_{32}, & \kappa q_{33}-a_{33}, & z \\ x, & y, & z, & 0 \end{vmatrix} = 0,$$

where, now, $$\kappa = (p_{11}x^2+2p_{12}xy+\ldots)\,\Delta/t^2$$
$$= (c_{11}x^2+2c_{12}xy+\ldots)/t^2.$$

On expanding the determinant, the equation to the wave surface may be put in the form

$$(c_{11}x^2+2c_{12}xy+\ldots)(d_{11}x^2+2d_{12}xy+\ldots)-t^2\Delta'(\xi-t^2) = 0,$$

where

$$\xi = (a_{22}q_{33}+a_{33}q_{22}-2a_{23}q_{23})\,x^2+(a_{13}q_{23}+a_{23}q_{13}-a_{12}q_{33}-a_{33}q_{12})\,xy+\ldots.$$

This is the equation to the most general wave surface possible in an æolotropic medium which satisfies the rotational condition and has two energy functions of the types postulated.

Since the equation to the surface in terms of X, Y, Z is a Fresnel's equation, and since X, Y, Z are linear functions of x, y, z, it follows that the most general wave surface (under the hypotheses stated) can be derived from a Fresnel's wave surface by applying a homogeneous strain. This theorem is given by Macdonald (at the end of his paper), without proof, for the case when Green's condition of complete transversality is satisfied.*

In order to compare the form of this wave surface with that given by Heaviside (*Electrical Papers*, Vol. II., pp. 1–23; reprinted from *Phil. Mag.*, Vol. XIX., Ser. 5, 1885, p. 397), I add a comparison of coefficients which will make the dynamical equations of the medium harmonize with the electromagnetic equations used by Heaviside. We need only write for my a_{rs} Heaviside's μ_{rs}; and for my q_{rs} his c_{rs}. It will follow that his scheme denoted by μ^{-1} has the coefficients p_{rs} of my notation; and his scheme c^{-1} has the coefficients d_{rs} of the work given above.

Using these facts, it will be seen that the wave surface is identical with the expanded form of Heaviside's [see equation (87) on p. 18 in the *Electrical Papers*]. For the completion of the identification, it may be added that his determinants m, n, are, respectively, Δ and $1/\Delta'$ of my notation.

4. *Application of the Results to deduce the Consequences of the Transversal Hypothesis.*

From the analogy already remarked between the two hypotheses (rotational and transverse) and the electromagnetic theory, it would

* This is proved, for the electromagnetic theory, by A. McAulay, *Phil. Mag.*, Vol. XLI., Ser. 5, 1896, p. 224.

be expected that the elastic displacements on the two hypotheses would be *linearly* related.*

We observe then, that the three equations

$$v^2 \frac{\partial A}{\partial a_r} = \frac{\partial B}{\partial a_r} \quad (r = 1, 2, 3)$$

are equivalent to the three

$$v^2 \frac{\partial A}{\partial \xi_r} = \frac{\partial B}{\partial \xi_r} \quad (r = 1, 2, 3),$$

provided that ξ_1, ξ_2, ξ_3 are three *independent linear* functions of a_1, a_2, a_3. We shall take, in fact,

$$\xi_r = \tfrac{1}{2} \frac{\partial A}{\partial a_r} = a_{r1} a_1 + a_{r2} a_2 + a_{r3} a_3 \quad (r = 1, 2, 3),$$

and observe that these three quantities are independent, since the determinant $|a_{rs}|$ cannot vanish (because A is a *definite* quadratic form). Expressing a_1, a_2, a_3 in terms of ξ_1, ξ_2, ξ_3, we have the equations

$$a_r = p_{r1}\xi_1 + p_{r2}\xi_2 + p_{r3}\xi_3 \quad (r = 1, 2, 3),$$

where the quantities p_{rs} are those defined in § 3.

Hence, expressing A in terms of ξ_1, ξ_2, ξ_3, we have

$$A = a_1\xi_1 + a_2\xi_2 + a_3\xi_3 = p_{11}\xi_1^2 + 2p_{12}\xi_1\xi_2 + \dots,$$

and, of course, B can be similarly expressed.

In terms of ξ_1, ξ_2, ξ_3, the transverse condition

$$la_1 + ma_2 + na_3 = 0$$

becomes

$$l(p_{11}\xi_1 + p_{12}\xi_2 + p_{13}\xi_3) + m(p_{21}\xi_1 + p_{22}\xi_2 + p_{23}\xi_3) \\ + n(p_{31}\xi_1 + p_{32}\xi_2 + p_{33}\xi_3) = 0,$$

or

$$l\frac{\partial A}{\partial \xi_1} + m\frac{\partial A}{\partial \xi_2} + n\frac{\partial A}{\partial \xi_3} = 0.$$

But this is of the same form, algebraically, as the condition $\gamma = 0$ found in our discussion of the rotational hypothesis (in § 2); in fact, we have only to replace a_r by ξ_r and a_{rs} by p_{rs} to make the two conditions identical, and the *same* substitutions connect the two expressions for A. Thus the final result of § 2 can be altered so as to apply in

* For the one case represents the elastic displacement by the magnetic force, and the other by the magnetic induction.

the present case by making the corresponding substitutions. In detail, we have to replace

e' by $l\xi_1 = l\,(a_{11}\alpha_1 + a_{12}\alpha_2 + a_{13}\alpha_3)$,

a' by $m\xi_3 + n\xi_2 = m\,(a_{31}\alpha_1 + a_{32}\alpha_2 + a_{33}\alpha_3) + n\,(a_{21}\alpha_1 + a_{22}\alpha_2 + a_{23}\alpha_3)$,

ϖ_1 by $m\xi_3 - n\xi_2 = m\,(a_{31}\alpha_1 + a_{32}\alpha_2 + a_{33}\alpha_3) - n\,(a_{21}\alpha_1 + a_{22}\alpha_2 + a_{23}\alpha_3)$.

Consequently we have to write (in the expression for V on p. 315)*

for e, $a_{11}\dfrac{\partial u_1}{\partial x} + a_{12}\dfrac{\partial u_2}{\partial x} + a_{13}\dfrac{\partial u_3}{\partial x} = \tfrac{1}{2}\left[2a_{11}e + a_{12}(c+\omega_3) + a_{13}(b-\omega_2)\right]$;

for a, $\left(a_{31}\dfrac{\partial u_1}{\partial y} + a_{32}\dfrac{\partial u_2}{\partial y} + a_{33}\dfrac{\partial u_3}{\partial y}\right) + \left(a_{21}\dfrac{\partial u_1}{\partial z} + a_{22}\dfrac{\partial u_2}{\partial z} + a_{23}\dfrac{\partial u_3}{\partial z}\right)$

$$= \tfrac{1}{2}\left[2a_{23}(f+g) + a_{21}(b+\omega_2) + a_{22}(a-\omega_1) + a_{31}(c-\omega_3) + a_{33}(a+\omega_1)\right];$$

for ω_1, $\left(a_{31}\dfrac{\partial u_1}{\partial y} + a_{32}\dfrac{\partial u_2}{\partial y} + a_{33}\dfrac{\partial u_3}{\partial y}\right) - \left(a_{21}\dfrac{\partial u_1}{\partial z} + a_{22}\dfrac{\partial u_2}{\partial z} + a_{23}\dfrac{\partial u_3}{\partial z}\right)$

$$= \tfrac{1}{2}\left[2a_{23}(f-g) - a_{21}(b+\omega_2) - a_{22}(a-\omega_1) + a_{31}(c-\omega_3) + a_{33}(a-\omega_1)\right],$$

with corresponding substitutions for the other six symbols; and, finally, the quantity

$$\Delta = p_{11}e + p_{12}c + \dots$$

is to be replaced by $(e+f+g)$.

When these substitutions are made in the value of V given at the end of § 2, the resulting expression will be the most general value of V for a medium which satisfies Green's condition of perfect transversality and is æolotropic as regards inertia.

As to the resulting wave surface, we have only to put p_{rs} in place of a_{rs} in the final equation given in § 3. This gives

$$(a_{11}x^2 + 2a_{12}xy + \dots)(d_{11}x^2 + 2d_{12}xy + \dots) - t^2\Delta\Delta'(\eta - t^2) = 0,$$

where

$$\eta = (p_{22}q_{33} + p_{33}q_{22} - 2p_{23}q_{23})\,x^2 + (p_{13}q_{23} + p_{23}q_{13} - p_{12}q_{33} - p_{33}q_{12})\,xy + \dots .$$

Just as before, this wave surface must be deducible from a Fresnel's surface by applying a homogeneous strain; and this is Macdonald's theorem.

* It may be remarked that in a medium isotropic as regards inertia we have

$$a_{11} = a_{22} = a_{33} = \rho, \quad a_{23} = a_{31} = a_{12} = 0,$$

and these three quantities reduce to ρe, ρa, $\rho\omega_1$, as they ought to do. For, in this case, the condition of § 2 is equivalent to the transversal condition; and so the final expression for V should be unchanged.

Thursday, March 13th, 1902.

Major MACMAHON, R.A., F.R.S., Vice-President, and subsequently Lt.-Col. CUNNINGHAM, R.E., in the Chair.

Thirteen members present.

Mr. Hardy was admitted into the Society.

The Rev. J. Cullen read a paper on "The Solutions of a System of Linear Congruences." Messrs. Mathews and Cunningham spoke on the subject of the paper.

Mr. Hardy gave an abstract of his paper "On the Theory of Cauchy's Principal Values (iii.)."

Mr. Hargreaves communicated a paper on "The Algebraical Connection between Zonal Harmonics of Orders differing by an Integer."

A paper on "Quadrature Formulæ," by Mr. J. Buchanan, was communicated from the Chair.

Dr. Macaulay gave an expression for the resultant of n homogeneous equations in n variables as a determinant divided by a minor of that determinant.

The following presents were made to the Library:—

"Educational Times," March, 1902.

"Indian Engineering," Vol. xxxi., Nos. 4–11, 1902.

"Mathematical Gazette," Vol. ii., No. 31, 1902.

"Periodico di Matematica," Serie 2, Vol. iv., Fasc. 4; Livorno.

"Supplemento al Periodico di Matematica," Anno v., Fasc. 4; Livorno, 1902.

Dickson, L. E.—"College Algebra," 8vo; New York, 1902.

Grossmann, Dr. E.—"Beobachtungen am Repsoldschen Meridiankreise," roy. 8vo; Leipzig, 1901.

Neumann, C.—"Ueber die Maxwell-Hertz'sche Theorie," roy. 8vo; Leipzig, 1901.

His, W.—"Beobachtungen zur Geschichte der Nasen- und Gaumenbildung beim menschlichen Embryo," roy. 8vo; Leipzig, 1901.

Dini, U.—"Sur la Méthode des Approximations successives pour les Equations aux Dérivées partielles du deuxième ordre," pamphlet, 4to; 1901.

"American Journal of Mathematics," Vol. xxiv., No. 1; January, 1902.

"Mathematisch-naturwissenschaftliche Mitteilungen von der Königl. Gesellschaft der Wissenschaften in Württemberg," Serie 2, Bd. iv., Heft 1; January, 1902.

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. LXIX., No. 456, 1902.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXVI., Heft 3; Leipzig, 1902.

"Annales de la Faculté des Sciences de Toulouse," Série 2, Tome III., Fasc. 2; Paris, 1901.

"Bulletin des Sciences Mathématiques," Tome XXVI., Jan., 1902; Paris.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. VIII., Fasc. 1: Napoli, 1902.

"Atti della Reale Accademia dei Lincei—Rendiconti." Sem. 1, Vol. XI., Fasc. 3, 4; Roma, 1902.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," Bd. LIII., Pts. 4-6; 1901.

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The Solutions of a System of Linear Congruences. By the Rev. J. Cullen, S.J. Received March 4th, 1902. Read March 13th, 1902.

The author is indebted to Lt.-Col. Allan Cunningham, R.E., not only for having carefully revised the original draft, but also for many important suggestions embodied in the paper.

1. The object of the present paper is to give a graphical process for obtaining solutions satisfying a system of linear congruences within a given limit. The process is adapted to the factorization of large composites, or the determination of large primes and other problems in the theory of numbers, such, for instance, as the representation of high numbers in binary quadratic forms, &c.

Incidentally some properties of a system of linear congruences are given that are believed to be new.

The scope of the paper consists in proving and explaining four simple rules to be employed in the application of the process, together with an example showing its working.

2. Let a quantity H be such that

$$\left.\begin{aligned} H &\equiv a_1,\ a_2,\ a_3,\ \dots,\ a_m \ (\text{mod } P) \\ &\equiv \beta_1,\ \beta_2,\ \beta_3,\ \dots,\ \beta_n \ (\text{mod } Q) \end{aligned}\right\} \qquad \text{(A)}$$

$$\left.\begin{aligned} &\equiv \rho_1',\ \rho_2',\ \rho_3',\ \dots,\ \rho_f' \ (\text{mod } p') \\ &\equiv \rho_1'',\ \rho_2'',\ \rho_3'',\ \dots,\ \rho_g'' \ (\text{mod } p'') \\ &\equiv \rho_1''',\ \rho_2''',\ \rho_3''',\ \dots,\ \rho_h''' \ (\text{mod } p''') \\ &\quad \vdots \quad \vdots \quad \vdots \qquad\quad \vdots \\ &\equiv \rho_1^{(\sigma)},\ \rho_2^{(\sigma)},\ \rho_3^{(\sigma)},\ \dots,\ \rho_s^{(\sigma)} \ (\text{mod } p^{(\sigma)}) \end{aligned}\right\}, \qquad \text{(B)}$$

where P and Q are prime or composite moduli, but prime to each other, $p', p'', p''', \dots, p^{(\sigma)}$ different odd primes, not contained in P or Q; $a_1, a_2, a_3, \dots, a_m$; $\beta_1, \beta_2, \dots, \beta_3, \dots, \beta_n$, and the ρ's least residues for the corresponding moduli. Our object then is to obtain integral solutions or values of H simultaneously satisfying these $\sigma+2$ congruences under a given limit of H.

3. We shall first combine the congruences (A) and give results on which the subsequent work depends. Now we know that on combination there arise mn cases with the modulus PQ, but, as will presently appear, we need only solve the n congruences

$$P\lambda + a_1 \equiv \beta_1,\ \beta_2,\ \beta_3,\ \dots,\ \beta_n \ (\text{mod } Q),$$

and the $m-1$ congruences

$$P\lambda + a_2,\ a_3,\ \dots,\ a_m \equiv \beta_1 \ (\text{mod } Q),$$

altogether $m+n-1$ instead of mn. The result of combining is, of course, that we may take $H = xPQ + r$, where r has mn values less than PQ. Hence we may denote any one of these mn values of r by $r_{\varpi,\kappa}$, since it is capable of representing mn values as ϖ ranges from 1 to m, and κ from 1 to n independently. Further, we may restrict $r_{\varpi,\kappa}$ to that case arising out of

$$P\lambda + a_\varpi \equiv \beta_\kappa \ (\text{mod } Q), \qquad (1)$$

observing that ϖ always refers to the subscript of one of the a's, and κ to that of one of the β's. It will also be useful to denote the corresponding value of λ by $\lambda_{\varpi,\kappa}$.

We have then $$H = xPQ + P\lambda_{\varpi,\kappa} + a_\varpi \qquad (2)$$

or $$r_{\varpi,\kappa} = P\lambda_{\varpi,\kappa} + a_\varpi, \qquad (3)$$

since $H \equiv a_{\varpi} \pmod{P} \equiv \beta_{\kappa} \pmod{Q}$ by (1) and (2); so we may take our solution to be

$$H = xPQ + r_{\varpi,\kappa}. \tag{4}$$

4. We now proceed to express (4) in another form. Let u be an integral solution (which is always possible) of

$$Qv - Pu = 1. \tag{5}$$

Then, multiplying by $a_{\varpi} - \beta_{\kappa}$, we have

$$Pu\,(a_{\varpi} - \beta_{\kappa}) \equiv \beta_{\kappa} - a_{\varpi} \pmod{Q}. \tag{6}$$

Hence, by (2), we see that

$$\lambda_{\varpi,\kappa} \equiv u\,(a_{\varpi} - \beta_{\kappa}) \pmod{Q}; \tag{7}$$

so that

$$\left.\begin{aligned} \lambda_{1,\kappa} &\equiv u\,(a_1 - \beta_{\kappa}) \\ \lambda_{\varpi,1} &\equiv u(a_{\varpi} - \beta_1) \\ \lambda_{1,1} &\equiv u\,(a_1 - \beta_1) \end{aligned}\right\} \pmod{Q}.$$

Therefore $\quad \lambda_{\varpi,1} + \lambda_{1,\kappa} - \lambda_{1,1} \equiv u\,(a_{\varpi} - \beta_{\kappa}) \equiv \lambda_{\varpi,\kappa} \pmod{Q}$

or

$$\lambda_{\varpi,\kappa} \equiv \lambda_{\varpi,1} + \lambda_{1,\kappa} - \lambda_{1,1} \pmod{Q}; \tag{8}$$

so that, on multiplying (8) by P and adding a_{ϖ}, we have

$$P\lambda_{\varpi,\kappa} + a_{\varpi} \equiv (P\lambda_{\varpi,1} + a_{\varpi}) + (P\lambda_{1,\kappa} + a_1) - (P\lambda_{1,1} + a_1) \pmod{PQ}$$

or, by (3),

$$r_{\varpi,\kappa} \equiv r_{\varpi,1} + r_{1,\kappa} - r_{1,1} \pmod{PQ}. \tag{9}$$

Hence, finally, we may take our solution to be

$$H = xPQ + r_{\varpi,1} + r_{1,\kappa} - r_{1,1}. \tag{10}$$

On this equation the whole process is based.

5. If we assign any value to ϖ between 1 and m inclusive, and ranges from 1 to n, we obtain the n quantities

$$r_{1,1} + (r_{\varpi,1} - r_{1,1}),\ r_{1,2} + (r_{\varpi,1} - r_{1,1}),\ r_{1,3} + (r_{\varpi,1} - r_{1,1}),\ \ldots,\ r_{1,n} + (r_{\varpi,1} - r_{1,1}),$$

together with a multiple of PQ by (10). We shall speak of these n quantities as belonging to the ϖ-*th arrangement*. The first arrangement consists simply of $r_{1,1}, r_{1,2}, r_{1,3}, \ldots, r_{1,n}$, together with the multiple of PQ. Hence it is important to observe that the ϖ-th arrangement is obtained from the first arrangement by the mere addition of the known quantity $(r_{\varpi,1} - r_{1,1})$.

6. The first step then in the application of the process which gives rise to Rule I. is to solve the n congruences

$$P\lambda+a_1\equiv\beta_1, \beta_2, \beta_3, \ldots, \beta_n \pmod Q$$

and the $m-1$ congruences

$$P\lambda+a_2, a_3, \ldots, a_m\equiv\beta_1 \pmod Q,$$

giving the cases $r_{1,1}, r_{1,2}, r_{1,3}, \ldots, r_{1,n}$ and $r_{2,1}, r_{3,1}, \ldots, r_{m,1}$ respectively, which values are easily found by applying (3), (5) and (7).

We next tabulate the *least* residues (θ) of these quantities for the prime moduli $p', p'', p''', \ldots, p^{\nu}$, and also the *least* residues (t) of PQ for the same moduli as follows:—

Modulus	$r_{1,1}$	$r_{1,2}$	$r_{1,3}$	......	$r_{1,n}$	$r_{2,1}$	$r_{3,1}$	......	$r_{m,1}$	PQ
p'	$\theta'_{1,1}$	$\theta'_{1,2}$	$\theta'_{1,3}$	......	$\theta'_{1,n}$	$\theta'_{2,1}$	$\theta'_{3,1}$	......	$\theta'_{m,1}$	t'
p''	$\theta''_{1,1}$	$\theta''_{1,2}$	$\theta''_{1,3}$	......	$\theta''_{1,n}$	$\theta''_{2,1}$	$\theta''_{3,1}$	......	$\theta''_{m,1}$	t''
p'''	$\theta'''_{1,1}$	$\theta'''_{1,2}$	$\theta'''_{1,3}$	......	$\theta'''_{1,n}$	$\theta'''_{2,1}$	$\theta'''_{3,1}$	......	$\theta'''_{m,1}$	t'''
$\vdots$	$\vdots$	$\vdots$	$\vdots$	......	$\vdots$	$\vdots$	$\vdots$	......	$\vdots$	$\vdots$
p^{ν}	$\theta^{\nu}_{1,1}$	$\theta^{\nu}_{1,2}$	$\theta^{\nu}_{1,3}$	......	$\theta^{\nu}_{1,n}$	$\theta^{\nu}_{2,1}$	$\theta^{\nu}_{3,1}$	......	$\theta^{\nu}_{m,1}$	t^{ν}

It will be useful to speak of this table as the *elements table*. The object of this table will be best understood when considered in conjunction with what follows and what has been already stated.

7. We shall now confine our attention for the moment to the first arrangement ($\varpi=1$) which gives us

$$H=xPQ+r_{1,\kappa},$$

κ ranging from 1 to n inclusive (since these numbers κ form the same set, viz., 1, 2, 3, ..., n in each arrangement, we shall speak of them as *set numbers*, and they will be denoted by κ), and we now proceed to show how values of x may be graphically obtained where

$$H=xPQ+r'_{1,\kappa}\equiv\rho'_d \pmod{p'},$$

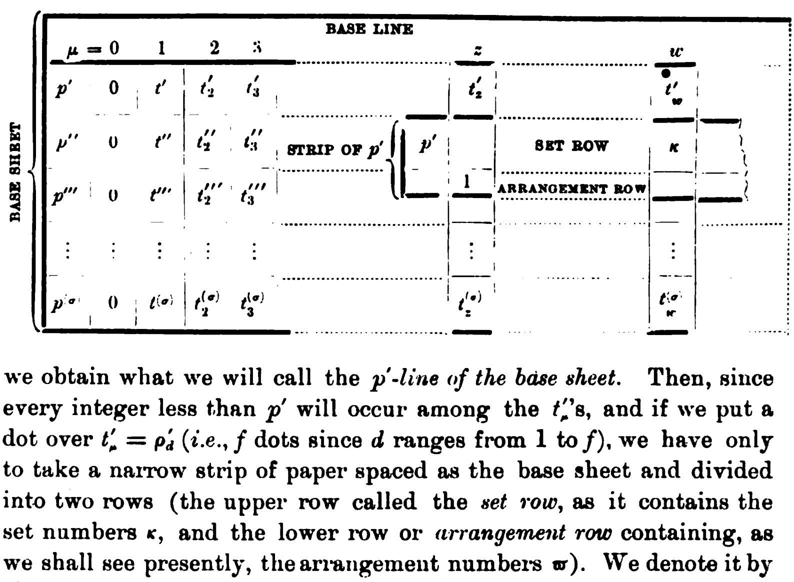
d ranging from 1 to f (*cf.* § 2). Now, from the elements table, we have

$$H\equiv xt'+\theta'_{1,\kappa}\equiv\rho'_d \pmod{p'}.$$

Let
$$\theta'_{1,\kappa} \equiv zt' \pmod{p'}$$
and
$$\rho'_d \equiv wt' \pmod{p'};$$
so that
$$H \equiv t'(x+z) \equiv wt' \pmod{p'} \quad \text{or} \quad x = w - z. \tag{11}$$

It is, however, quite needless in practice to solve these congruences, for, if we take a sheet of paper with vertical rules at *equal* horizontal spacing, and write down the *least* residues t'_μ of the successive multiples of t' for the modulus p' as follows with

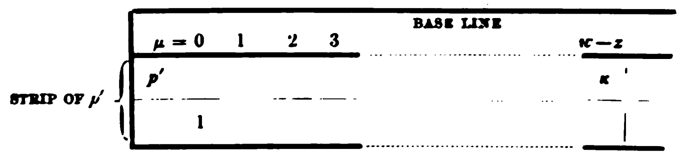
$$t'_\mu \equiv \mu t' \pmod{p'},$$



we obtain what we will call the *p′-line of the base sheet.* Then, since every integer less than p' will occur among the t'_μ's, and if we put a dot over $t'_\mu = \rho'_d$ (*i.e.*, f dots since d ranges from 1 to f), we have only to take a narrow strip of paper spaced as the base sheet and divided into two rows (the upper row called the *set row*, as it contains the set numbers κ, and the lower row or *arrangement row* containing, as we shall see presently, the arrangement numbers ϖ). We denote it by $\acute{p}$, and we write 1 in the arrangement row to give the initial division of the first arrangement. Then, if we place the strip with this division at t'_z ($= \theta'_{1,\kappa}$) of the p'-line, and write the values of κ in the divisions of the set-row, under the dotted figures (*i.e.*, t'_w), we obtain the values of x by simply placing the strip under the *base line** so that the initial division marked 1 is at the 0 (or any multiple of p') of the base line, and read off the integers in the base line that are over the κ's, since, if we compare the preceding figure with the

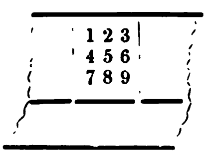
* The *base line* is the top line of the base sheet, and consists merely of successive integers in each division; its use is to give the value of x as explained above.

following, we see at once that the number over κ is $\varpi - z$, which is the required value for x by (11).



8. In applying the principle just described, it is evident that as we placed the initial division of the strip at t'_z $(= \theta'_{1,\kappa})$, and wrote κ under each dotted figure, so we first place the initial division at t'_μ $(= \theta'_{1,1})$, and write 1 in the set row under each dotted figure; then it is placed at t'_μ $(= \theta'_{1,2})$, and 2 is written in the same row under each dotted figure and so on till t'_μ $(= \theta'_{1,n})$ has been dealt with and n written.

In filling up the divisions in this manner, it is necessary that each number should always occupy a fixed relative position with respect to the other set numbers: *e.g.*, if $n = 9$, then we should follow the scheme



so that 3, for instance, should always occupy the right-hand top corner in those divisions in which it is to appear.

9. In the last paragraph we have shown how the strip for the prime p' is to be drawn up, and in a precisely similar manner strips are drawn up for the primes $p'', p''', \ldots, p^{(\sigma)}$. All the *least* residues (t'_μ) of the successive multiples of PQ are arranged in the base sheet as shown in the first figure, § 7, and dots are placed over the t_μ's of each line that equal the ρ's of the corresponding congruence of the system (B). It should be noticed that the residues t'_μ recur at intervals of p', t''_μ at intervals of p'', and so on. Hence the base sheet may be of any convenient length so long as the number of divisions in it exceeds $p^{(\sigma)}$, the highest prime. A similar remark applies to the strips.

10. So far we have been dealing only with the first arrangement, and the question that now arises is whether the strips already drawn

up for this arrangement suit for the other arrangements. We will now show that they do suit.

Taking the ϖ-th arrangement and attending to the remark in § 5, we have, by (10),

$$H = xPQ + r_{1,\kappa} + (r_{\varpi,1} - r_{1,1}) \equiv \rho'_d \pmod{p'}. \qquad (12)$$

Now, let $$(r_{\varpi,1} - r_{1,1}) \equiv yt' \pmod{p'}.$$

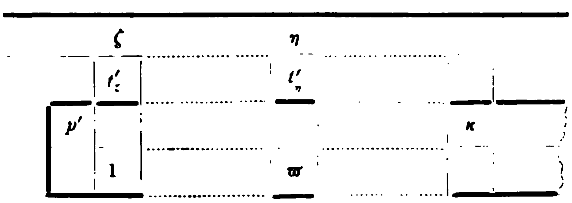
Then, by definition, y is constant for this arrangement and independent of w and z. Hence, as in § 7, we have

$$(x+z+y)\,t' \equiv wt' \pmod{p'},$$

or $$x = w - z - y\,; \qquad (13)$$

but $w-z$ is the distance between the initial division and κ in the first arrangement, and, by (13), we see that for the ϖ-th arrangement the effect of adding the quantity $(r_{\varpi,1} - r_{1,1})$ is only to shorten or lengthen the distance between the initial division of the first arrangement and that containing κ by y divisions, and further, since κ stands successively for the set of numbers 1, 2, 3, ..., n (*cf.* § 8), we see that all these numbers are affected in the same manner by the addition of $(r_{\varpi,1} - r_{1,1})$ to the first arrangement; hence we have only to find a new initial division for each arrangement.

11. It is here, then, that we make use of the second portion of the elements table, § 6, namely, the residues (θ) of $r_{2,1}, r_{3,1}, \ldots, r_{m,1}$. For, if we were to draw up a slip for the ϖ-th arrangement for the prime p', we should place its initial division (*i.e.*, the division having ϖ in the arrangement row of the strip) at the number $t'_\eta \,(= \theta'_{\varpi,1})$ of the base sheet, and write 1 in the set row under each dotted figure; but it is quite clear that, if we take the strip of p' already drawn up, the same result is obtained by placing the initial division (marked 1 in the arrangement row) at $t'_\zeta \,(= \theta'_{1,1})$ and writing ϖ in the arrangement row of that division, that is under $t'_\eta \,(= \theta'_{\varpi,1})$, since 1 of the set row is under each dotted figure. In fact, if we refer to the following figure, it is clear that the distance (a) between ϖ and κ is equal to



the distance (b) between 1 and κ less the distance (c) between 1 and ϖ; but

$$b = w - z,$$

by (11), and

$$c = \eta - \zeta = y,$$

since, by § 10.

$$r_{\varpi,1} - r_{1,1} \equiv yt' \pmod{p'} \equiv \theta_{\varpi,1} - \theta_{1,1} \equiv t'(\eta - \zeta):$$

therefore

$$a = w - z - y;$$

so that, if the strip of p' be placed with the division containing ϖ at the 0 of the base line, the number in the base line over κ would be $(w - z - y)$ which, by (13), is the value of x in

$$H = xPQ + r_{1,\kappa} + (r_{\varpi,1} - r_{1,1}) \equiv \rho'_d \pmod{p'}.$$

Thus we obtain our general solutions.

Hence the new initial divisions for the other arrangements are obtained by placing the strip p' so that the division containing 1 in the arrangement row is under t_1 $(= \theta'_{1,1})$ of the p' line of the base sheet, and writing 2, 3, ..., m in the arrangement row of the strip under the divisions (of the p' line of the base sheet) containing the numbers $t'_{\varpi,1}$ $(= \theta'_{2,1}, \theta'_{3,1}, \ldots, \theta'_{m,1}$ of the elements table).

12. Similarly the initial divisions for the different arrangements for the strips p'', p''', ..., $p^{(\sigma)}$ are obtained from their respective rows in the elements table and the base sheet.

13. We are now in a position to obtain values of x in (10) simultaneously satisfying the $\sigma + 2$ congruences (A) and (B), supposing the σ strips to have been completed; also the values of ϖ and κ can now be found, and hence, by (10), and the top row of the elements table, our solutions can be obtained. For, if we place all the strips one under the other under the base line, so that the initial divisions containing ϖ in the arrangement rows form a column under 0,* and if we search the columns for a number κ appearing in the *set rows* of *all* the strips in a particular column, and read off the number

* Any strip for $p^{(r)}$ (say) may be placed so that the division with ϖ in its arrangement row is under any multiple of $p^{(r)}$ in the base line.

x in the base line over this column, a solution is

$$xPQ+r_{\varpi,1}+r_{1,\kappa}-r_{1,1},$$

while, if κ fails to appear in *any* strip, this cannot be a solution, as is clear from § 7.

Thus, in practice, we begin by placing all the strips so that the initial divisions containing 1 in the arrangement row of each are at the 0 of the base line; we then search up to the required limit of H; then we place the divisions containing 2 in the arrangement row of each strip at 0 of the base line, and continue the search; and so on till m has been dealt with.

If L be the upper limit of H, then $x \ngtr L/PQ$; yet we should search to the $(L/PQ+1)$-th column, since $(r_{\varpi,1}+r_{1,\kappa}-r_{1,1})$ may be negative, but $<PQ$; also this quantity may be $>PQ$, but $<2PQ$. Therefore x may $=-1$, and yield a positive solution. Thus the column of the strips to the left of the 0 of the base line should be searched.

14. Rules.—*Rule I.*—Apply (3), (5), and (7) to solve the n congruences

$$P\lambda+a_1 \equiv \beta_1, \beta_2, \beta_3, \dots, \beta_n \pmod{Q},$$

and the $m-1$ congruences

$$P\lambda+a_2, a_3, \dots, a_m \equiv \beta_1 \pmod{Q},$$

giving the cases $r_{1,1}, r_{1,2}, r_{1,3}, \dots, r_{1,n}$ and $r_{2,1}, r_{3,1}, \dots, r_{m,1}$, respectively. Then form the elements table, § 6.

Rule II.—Form the base sheet, § 7, and in the line p' place dots over the numbers $t'_\mu = \rho'_1, \rho'_2, \dots, \rho'_f$, and so in like manner treat the lines $p'', p''', \dots, p^{(\sigma)}$; then place the initial division (marked 1 in the arrangement row) of the strip p' at t'_{ι_1} (of the base sheet) $= \theta'_{1,1}$ (of the elements table), and write 1 under each dotted figure; then place the same division at $t'_{\iota_2} = \theta'_{1,2}$, and write 2 under each dotted figure; and so on till m has been written in the set row of the strip. Thus, in a similar manner, complete the strips for $p'', p''', \dots, p^{(\sigma)}$.

Rule III. — Now place the initial division (with 1 in the arrangement row) at $t'_{\iota_1} = \theta'_{1,1}$, as in Rule II., but now write $2, 3, \dots, m$ in the arrangement row in the divisions under

$$t'_\varpi = \theta'_{2,1}, \theta'_{3,1}, \dots, \theta'_{m,1}.$$

Thus also find the initial divisions of the different arrangements for the strips p'', p''', ..., $p^{(\sigma)}$.*

Rule IV.—Place all the strips one under the other so that the initial divisions (marked 1 in the arrangement rows) form a column under the 0 of the base line, and search for a number appearing in the *set row* of *every* strip throughout a particular column; having thus searched throughout $(L/PQ+2)$ columns ($x = -1$ to $x = L/PQ+1$), we place the strips with 2 of the arrangement rows under the 0 of the base line and continue the search, and so on till m of all the arrangement rows has been placed under the 0, and all the columns searched. If, then, when ϖ of the arrangement rows is under the 0 (or any multiple of $p^{(\tau)}$ for the strip $p^{(\tau)}$) of the base line, and the number κ appears throughout the divisions of the set rows which form a column under x of the base line, then $xPQ+r_{\varpi,1}+r_{1,\kappa}-r_{1,1}$ is a solution.

15. We now give an example showing the working of the process. Let $N = 1{,}886{,}601{,}653$, and, if we wish to determine whether N be prime or composite, we may seek the partition $N = H^2+G^2$, since $N = 4k+1$: if this partition be unique, N is prime, while, if there is no partition or else two or more partitions, then N is composite. Taking H to be odd, we have $H^2 = N-G^2$. Now

$$N \equiv 2 \pmod{3} \equiv 3 \pmod{5} \equiv 6 \pmod{7} \equiv 53 \pmod{64};$$

therefore

$$H \equiv \pm 1 \pmod{3} \equiv \pm 2 \pmod{5} \equiv \pm 2 \pm 3 \pmod{7} \equiv \pm 7 \pm 9 \pmod{32}.$$

We combine $H \equiv \pm 1 \pmod{3} \equiv \pm 2 \pm 3 \pmod{7}$,

giving $$H \equiv \pm 2 \pm 4 \pm 5 \pm 10 \pmod{21},$$

and also $$H \equiv \pm 2 \pmod{5} \equiv \pm 7 \pm 9 \pmod{32},$$

giving $$H \equiv \pm 7 \pm 23 \pm 57 \pm 73 \pmod{160}.$$

* In the application of Rule II. for finding the set numbers κ, and of Rule III. for finding the arrangement numbers ϖ, it should be noticed that the $(t^{(\tau)})$ may be to the left of the initial division, but, if a second 1 be written in the arrangement row of the strip $p^{(\tau)}$ at a distance of $p^{(\tau)}$ divisions from the initial division, we have only to move the strip so that the latter division occupies the place of the former to give us a division in which κ or ϖ is to be written.

Hence, taking $P = 160$ and $Q = 21$, we have

$$
\left.
\begin{array}{l}
H \equiv \left.\begin{array}{l} \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7,\ \alpha_8 \\ 7,\ 23,\ 57,\ 73,\ 87,\ 103,\ 137,\ 153 \end{array}\right\} \pmod{160} \\
\quad \equiv \left.\begin{array}{l} \beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \beta_5,\ \beta_6,\ \beta_7,\ \beta_8 \\ 2,\ 4,\ 5,\ 10,\ 11,\ 16,\ 17,\ 19 \end{array}\right\} \pmod{21}
\end{array}
\right\} \qquad \text{(A)}
$$

$$
\left.
\begin{array}{ll}
\equiv 1, 2, 3, 8, 9, 10 \pmod{11} & \text{since } N \equiv 2 \pmod{11} \\
\equiv 0, 1, 2, 6, 7, 11, 12 \pmod{13} & \equiv 1 \pmod{13} \\
\equiv 1, 4, 7, 8, 9, 10, 13, 16 \pmod{17} & \equiv 14 \pmod{17} \\
\equiv 0, 3, 4, 7, 8, 9, 10, 11, 12, 15, 16 \pmod{19} & \equiv 16 \pmod{19} \\
\equiv 1, 4, 6, 7, 8, 11, 12, 15, 16, 17, 19, 22 \pmod{23} & \equiv 19 \pmod{23} \\
\equiv 4, 5, 6, 8, 10, 11, 12, 17, 18, 19, 21, 23, & \\
\qquad 24, 25 \pmod{29} & \equiv 12 \pmod{29} \\
\equiv 1, 4, 5, 7, 9, 10, 14, 15, 16, 17, 21, 22, & \\
\qquad 24, 26, 27, 30 \pmod{31} & \equiv 26 \pmod{31} \\
\equiv 2, 4, 5, 6, 9, 12, 13, 14, 18, 19, 23, 24, & \\
\qquad 25, 28, 31, 32, 33, 35 \pmod{37} & \equiv 32 \pmod{37}
\end{array}
\right\}.
$$

(B)

Now $21v - 160u = 1$ gives $u = 8$; hence, applying (7) and then (3), we find

	$r_{1,1}$	$r_{1,2}$	$r_{1,3}$	$r_{1,4}$	$r_{1,5}$	$r_{1,6}$	$r_{1,7}$	$r_{1,8}$	$r_{2,1}$	$r_{3,1}$	$r_{4,1}$	$r_{5,1}$	$r_{6,1}$	$r_{7,1}$	$r_{8,1}$	PQ
Modulus.	3047	487	2567	2887	1607	1927	647	1447	23	3257	233	1367	1703	1577	1913	3360
11	0	3	4	5	1	2	9	6	1	1	2	3	9	4	10	5
13	5	6	6	1	8	3	10	4	10	7	12	2	0	4	2	6
17	4	11	0	14	9	6	1	2	6	10	12	7	3	13	9	11
19	7	12	2	18	11	8	1	3	4	8	5	18	12	0	13	16
23	11	4	14	12	20	18	3	21	0	14	3	10	1	13	4	2
29	2	23	15	16	12	13	9	26	23	9	1	4	21	11	28	25
31	9	22	25	4	26	5	27	21	23	2	16	3	29	27	22	12
37	13	6	14	1	16	3	18	4	23	1	11	35	1	23	26	30

This is the elements table of § 6 and application of Rule I.

On the accompanying diagram we have the base sheet and the strips drawn up by Rules II. and III. They are in position for the first arrangement $\varpi = 1$, and we search the set rows in each column for a number appearing throughout up to the fourteenth column. Since $H \not> \sqrt{N} \not> 43435$ and $PQ = 3360$, which is the application of Rule IV., we find

x	6	10	12	13	1	6	0
ϖ	1	1	1	2	5	5	6
κ	6	3	3	5	8	5	3
$H =$	22087	36167	42887	42263	3127	20087	1223

The first three results $\varpi = 1$ are shown on the strips, viz., 6 appears in set rows in the column under 6, of the base line, and 3 in the columns under 10 and 12. On actual trial we have

$$N = 1{,}886{,}601{,}653 = 42887^2 + 6878^2 = 42263^2 + 10022^2$$
$$= 17837 \,.\, 105769.$$

The other five solutions would soon fail to conform to subsequent moduli if we used strips for 41, 43, &c.

Note (i.).—In searching any particular column we compare the numbers of any two adjacent set rows in the column and mentally carry the numbers common to the two rows into a third, and then those common to the three rows into a fourth, and so on. Thus, for instance, in the column under 7 of the base line the two top strips give 5 and 8 common, while 5 is only common to the third; so we need only look for 5 in the remaining set rows of the column, and, since 5 does not appear in strip 29, we pass on to the next column, 8.

Note (ii.).—It is well to observe the advantage of the graphical work over purely arithmetical, for each column deals with eight cases (in the example). Thus the column under 4 of the base line gives the eight cases of the first part of the elements table $+ 4.3360$. Then in arithmetical work we should have to find their residues to (mod 11), then those of the five cases $r_{1,1}, r_{1,2}, r_{1,3}, r_{1,4}, r_{1,5} + 4.3360$ to (mod 13), then $r_{1,4}, r_{1,5} + 4.3360$ to (mod 17), and, finally, $r_{1,4}$ to (mod 19), in order to exclude these eight cases, in all $8 + 5 + 2 + 1 = 16$ residues; but in graphical work a glance at the column gives this information.

5	26	27	28	29	30	31	32	33	34	35	36	37
6	12	8	4	0								
31	2	14	26	7	19	0						
10	3	33	26	19	12	5	35	28	21	14	7	0

5	26	27	28	29	30	31	32	33	34	35	36	37
	2 5 6 7	1 3 4 6 7 8	1 3 6	2 5								
				1								
3 6	2 3 6	2 5 7 8	1 2 4 5 7 8	1 3	2 5 6 7 8	1 2 4 5 6 7 8						
		2		4		1						
3 6	2 4 5 6	1 2 5 7	1 2 5	1 2 3 5 8	1 2 4 5	1 3 4 7 8	2 3 5 8	1 3 6 7 8	3 6 7 8	3 7 8	2 5 7	1 2 3 7 8
							4					1

The Errors in certain Quadrature Formulæ. *By* J. BUCHANAN, M.A., F.I.A. Received March 5th, 1902. Read March 13th, 1902. Received, in revised form,* May 18th, 1902.

1. In a recent paper† Mr. Sheppard has derived from the Maclaurin summation theorem many of the best known quadrature formulæ, with expressions for the errors in terms of differential coefficients. The use of differential coefficients, however, is inconvenient in cases where we do not know the form of the function, but only its value at stated intervals; and these are cases where the formulæ are of great practical value. Prof. Everett has recently given a new interpolation formula‡ involving only even central differences; and from it similar quadrature formulæ can be obtained by direct integration with expressions for the errors in terms of central differences.

Denoting by p the distance of the ordinate u_p in front of u_0, and by q its distance behind u_1, so that $p+q=1$, he writes

$$u_p = \left[q+\frac{q(q^2-1)}{3!}\,\delta^2+\frac{q(q^2-1)(q^2-4)}{5!}\,\delta^4+\dots\right]u_0$$

$$+\left[p+\frac{p(p^2-1)}{3!}\,\delta^2+\frac{p(p^2-1)(p^2-4)}{5!}\,\delta^4+\dots\right]u_1, \qquad (1)$$

where§ $$\delta u_0 = u_{\frac{1}{2}}-u_{-\frac{1}{2}},$$

so that δ^2, δ^4, ... are the even central differences.

If we integrate with respect to p between limits 0 and 1, we have, since $dq = -dp$,

$$\int_0^1 u_p\,dp = \int_0^1\left[p+\frac{p(p^2-1)}{3!}\,\delta^2+\frac{p(p^2-1)(p^2-4)}{5!}\,\delta^4+\dots\right](u_0+u_1)\,dp$$

$$= \tfrac{1}{2}\left[1-\tfrac{1}{12}\delta^2+\tfrac{11}{720}\delta^4-\tfrac{191}{60480}\delta^6+\tfrac{2497}{3628800}\delta^8-\dots\right](u_0+u_1)$$

$$= \tfrac{1}{2}\left[1+\lambda_2\delta^2+\lambda_4\delta^4+\lambda_6\delta^6+\dots\right](u_0+u_1),$$

where λ_2, λ_4, ... are written for shortness for the numerical coefficients.

* [The title of the paper was changed in revision: *cf. Proceedings*, *supra*, p. 322.—SEC.]

† *Proceedings*, Vol. XXXII., p. 258.

‡ *Journal of the Institute of Actuaries*, Vol. XXXV., p. 452.

§ *Cf. Proceedings*, Vol. XXXI., pp. 459–60.

Hence
$$\int_0^n u_p dp = \int_0^1 u_p dp + \int_1^2 u_p dp + \ldots + \int_{n-1}^n u_p dp$$
$$= (1+E+E^2+\ldots+E^{n-1}) \int_0^1 u_p dp$$
$$= \tfrac{1}{2}u_0 + u_1 + u_2 + \ldots + \tfrac{1}{2}u_n$$
$$+ (\lambda_2\delta + \lambda_4\delta^3 + \ldots)\, \mu\, (u_n - u_0), \qquad (2)$$

where*
$$\mu u_0 = \tfrac{1}{2}(u_{\frac{1}{2}} + u_{-\frac{1}{2}}),$$

so that $\mu\delta$, $\mu\delta^3$, … are the odd central differences.

2. Let $a, b, c, \ldots$ be the factors, including unity, of n; and let $\mu_a\delta_a$, $\mu_a\delta_a^3$, … be the odd central differences of the series of functions $\ldots u_{-2ah}, u_{-ah}, u_0, u_{ah}, u_{2ah}, \ldots$; then

$$\int_0^{nh} u_x dx = ah\left[\tfrac{1}{2}u_0 + u_{ah} + u_{2ah} + \ldots + \tfrac{1}{2}u_{nh}\right]$$
$$+ ah\left[\lambda_2\delta_a + \lambda_4\delta_a^3 + \ldots\right]\mu_a\,(u_{nh} - u_0).$$

Now
$$\mu\delta^{2n-1} = \cosh \tfrac{1}{2}hD\,(2\sinh \tfrac{1}{2}hD)^{2n-1},$$
$$\mu_a\delta_a^{2n-1} = \cosh \tfrac{1}{2}ahD\,(2\sinh \tfrac{1}{2}ahD)^{2n-1},$$

and since†

$$2\sinh a\phi\cosh a\phi = a\cosh\phi\left[2\sinh\phi + \frac{a^2-1}{3!}(2\sinh\phi)^3\right.$$
$$+\frac{(a^2-1)(a^2-4)}{5!}(2\sinh\phi)^5$$
$$\left.+\frac{(a^2-1)(a^2-4)(a^2-9)}{7!}(2\sinh\phi)^7+\ldots\right]$$

$$(2\sinh a\phi)^3\cosh a\phi = a^3\cosh\phi\left[(2\sinh\phi)^3 + \frac{a^2-1}{4}(2\sinh\phi)^5\right.$$
$$\left.+\frac{(a^2-1)(3a^2-7)}{4.5.6}(2\sinh\phi)^7+\ldots\right]$$

$$(2\sinh a\phi)^5\cosh a\phi = a^5\cosh\phi\left[(2\sinh\phi)^5 + \frac{a^2-1}{3}(2\sinh\phi)^7+\ldots\right],$$

… … … … … … … … … …

* *Proceedings*, Vol. xxxi., pp. 459-60.
† *Ibid.*, Vol. xxxi., p. 454.

we have at once

$$\begin{aligned}
\mu_a\delta_a &= a\mu\left[\delta+\frac{a^2-1}{3!}\,\delta^3+\frac{(a^2-1)(a^2-4)}{5!}\,\delta^5+\frac{(a^2-1)(a^2-4)(a^2-9)}{7!}\,\delta^7+\ldots\right],\\
\mu_a\delta_a^3 &= a^3\mu\left[\delta^3+\frac{a^2-1}{4}\,\delta^5+\frac{(a^2-1)(3a^2-7)}{4.5.6}\,\delta^7+\ldots\right]\\
\mu_a\delta_a^5 &= a^5\mu\left[\delta^5+\frac{a^2-1}{3}\,\delta^7+\ldots\right],\\
\mu_a\delta_a^7 &= a^7\mu\left[\delta^7+\ldots\right].\\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned}$$

If we make these substitutions for $\mu_a\delta_a$, $\mu_a\delta_a^3$, ..., and replace $\lambda_2, \lambda_4, \ldots$ by their numerical values, equation (2) reduces to

$$\begin{aligned}
A = \int_0^{nh} u_x\,dx = A_a - \frac{a^2h}{720}\Big[6\delta-(a^2+10)\,\delta^3+\left(\frac{a^4}{42}+\frac{a^2}{4}+2\right)\delta^5 \\
-\left(\frac{a^6}{1680}+\frac{a^4}{126}+\frac{7a^2}{120}+\frac{3}{7}\right)\delta^7+\ldots\Big]\,\mu\,(u_{nh}-u_0), \quad (3)
\end{aligned}$$

where $\qquad A_a = ah\left[\tfrac{1}{2}u_0+u_{ah}+u_{2ah}+\ldots+\tfrac{1}{2}u_{nh}\right].$

An approximate expression, together with an expression for the error in terms of differences, will be got by writing

$$(p+q+r+\ldots)A = pA_a+qA_b+rA_c+\ldots,$$

where p, q, r, ... are chosen to make the coefficients of the successive differences vanish.

The formulæ which can be got in this way are identical with those of Mr. Sheppard's paper,* as might be expected if we observe that the principal equation (2) obtained above is the central difference equivalent of the Maclaurin summation theorem. The method is, however, capable of extension.

3. The group of formulæ obtained by putting $n = 6$ is of special interest, as it includes a large number of those best known, and it is proposed to discuss it here in some detail. In what follows the letter h is for convenience omitted from the suffix of u.

If we choose p and q so that $pa^2+qb^2 = 0$, and put $a = 1$, $b = 2$, we get

$$A = \frac{h}{3}\,[4A_1-A_2] = \frac{h}{3}\left[u_0+u_6+4\,(u_1+u_3+u_5)+2\,(u_2+u_4)\right], \quad \text{(i.)}$$

* *Proceedings*, Vol. xxxii., pp. 262-65.

which is Simpson's rule, with an error

$$-\frac{h}{180}\left[\delta^3-\tfrac{31}{84}\delta^5+\tfrac{557}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0).$$

Putting $a=1$, $b=3$, so that $p:q::9:-1$, we have

$$A=\frac{h}{8}\,[9A_1-A_3]=\frac{3h}{8}\,[u_0+u_6+3\,(u_1+u_2+u_4+u_5)+2u_3], \qquad \text{(ii.)}$$

which is Simpson's second rule, with an error

$$-\frac{h}{80}\left[\delta^3-\tfrac{41}{84}\delta^5+\tfrac{987}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0).$$

Put $a=1:b=6$; then $p:q::36:-1$, and

$$A=\frac{h}{35}\,[36A_1-A_6]=\frac{h}{35}\,[15\,(u_0+u_6)+36\,(u_1+u_2+\dots+u_5)], \qquad \text{(iii.)}$$

with an error $-\dfrac{h}{20}\left[\delta^3-\tfrac{95}{84}\delta^5+\tfrac{5773}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0).$

Other formulæ involving only five terms can be obtained from the above by elimination. Thus the elimination of u_0 and u_6 between (i.) and (ii.) gives

$$A=h\,[3A_1-3A_2+A_3]=3h\,[u_1+u_5-(u_2+u_4)+2u_3], \qquad \text{(iv.)}$$

with an error $+\dfrac{h}{20}\left[\delta^3-\tfrac{51}{84}\delta^5+\tfrac{1377}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0).$

The elimination of u_1 and u_5 between the same two formulæ gives

$$A=\frac{h}{5}\,[9A_2-4A_3]=\frac{3h}{5}\,[u_0+u_6+6\,(u_2+u_4)-4u_3], \qquad \text{(v.)}$$

with an error $-\dfrac{h}{20}\left[\delta^3-\tfrac{47}{84}\delta^5+\tfrac{1213}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0).$

The result of eliminating u_2 and u_4 between (i.) and (ii.) is

$$A=\frac{h}{11}\,[18A_1-9A_2+2A_3]=\frac{3h}{11}\,[u_0+u_6+6\,(u_1+u_5)+8u_3], \qquad \text{(vi.)}$$

with an error $+\dfrac{h}{220}\left[\delta^3-\tfrac{71}{84}\delta^5+\tfrac{2197}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0),$

while the elimination of u_3 gives

$$A=\frac{h}{7}\,[6A_1+3A_2-2A_3]=\frac{3h}{7}\,[u_0+u_6+2\,(u_1+u_5)+4\,(u_2+u_4)], \qquad \text{(vii.)}$$

with an error $-\dfrac{h}{140}\left[3\delta^3-\tfrac{133}{84}\delta^5+\tfrac{3311}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0).$

Several of the above will be found to give very good results considering the small number of terms used and the simplicity of the coefficients.

4. From the above we can get other formulæ which are true to fifth differences. Thus, if we eliminate δ^3 between (i.) and (ii.), we get

$$A = \frac{h}{10}[15A_1 - 6A_2 + A_3] = \frac{3h}{10}[u_0 + u_2 + u_4 + u_6 + 5(u_1 + u_5) + 6u_3], \quad \text{(viii.)}$$

which is Weddle's rule, with an error

$$-\frac{h}{840}[\delta^5 - \tfrac{41}{60}\delta^7 + \dots]\,\mu\,(u_6 - u_0).$$

The elimination of δ^3 between (i.) and (iii.) gives

$$A = \frac{h}{280}[384A_1 - 105A_2 + A_6]$$

$$= \frac{h}{140}[45(u_0 + u_6) + 192(u_1 + u_3 + u_5) + 87(u_2 + u_4)], \quad \text{(ix.)}$$

with an error $$-\frac{h}{210}[\delta^5 - \tfrac{163}{120}\delta^7 + \dots]\,\mu\,(u_6 - u_0),$$

while the elimination of δ^3 between (ii.) and (iii.) gives

$$A = \frac{h}{210}[243A_1 - 35A_3 + 2A_6]$$

$$= \frac{h}{70}[25(u_0 + u_6) + 81(u_1 + u_2 + u_4 + u_5) + 46u_3], \quad \text{(x.)}$$

with an error $$-\frac{9h}{840}[\delta^5 - \tfrac{801}{540}\delta^7 + \dots]\,\mu\,(u_6 - u_0).$$

As before, we can, by elimination, get other formulæ involving only five terms: thus, if we eliminate u_0 and u_6 between (viii.) and (ix.), we get

$$A = \frac{h}{20}[66A_1 - 75A_2 + 30A_3 - A_6]$$

$$= \frac{3h}{10}[11(u_1 + u_5) - 14(u_2 + u_4) + 26u_3], \quad \text{(xi.)}$$

with an error $$h\,[\tfrac{41}{840}\delta^5 - \tfrac{3949}{50400}\delta^7 + \dots]\,\mu\,(u_6 - u_0).$$

The elimination of u_1 and u_5 between the same two formulæ gives

$$A = \frac{h}{120}[243A_2 - 128A_3 + 5A_6]$$

$$= \frac{h}{20}[11(u_0+u_6) + 81(u_2+u_4) - 64u_3], \qquad \text{(xii.)}$$

with an error $\quad -h\left[\frac{36}{840}\delta^5 - \frac{561}{8400}\delta^7 + \ldots\right]\mu(u_6 - u_0)$,

while the elimination of u_2 and u_4 gives

$$A = \frac{h}{300}[486A_1 - 243A_2 + 58A_3 - A_6]$$

$$= \frac{h}{50}[14(u_0+u_6) + 81(u_1+u_5) + 110u_3],^* \qquad \text{(xiii.)}$$

with an error $\quad h\left[\frac{3}{1400}\delta^5 - \frac{1}{224}\delta^7 + \ldots\right]\mu(u_6 - u_0)$.

By eliminating δ^5 between any two of these fifth difference formulæ, we get

$$A = \frac{h}{840}[1296A_1 - 567A_2 + 112A_3 - A_6]$$

$$= \frac{h}{140}[41(u_0+u_6) + 216(u_1+u_5) + 27(u_2+u_4) + 272u_3], \qquad \text{(xiv.)}$$

with an error $\quad -\frac{3h}{2800}\mu\delta^7(u_6 - u_0) + \ldots$.

5. Of the preceding formulæ some err in excess, and others in defect, of the true value, and by combining them in various ways the error can often be considerably reduced. For example, by taking the mean of (iv.) and (v.), both of which show a relatively large error, we get Weddle's rule. If we take the mean of (i.) and (vi.), we get

$$A = \frac{h}{33}[49A_1 - 19A_2 + 3A_3]$$

$$= \frac{h}{33}[10(u_0+u_6) + 49(u_1+u_5) + 11(u_2+u_4) + 58u_3], \qquad \text{(xv.)}$$

with an error $\; -\frac{h}{1980}\left[\delta^3 + \frac{149}{84}\delta^5 + \frac{6823}{5040}\delta^7 - \ldots\right]\mu(u_6 - u_0)$,

* *Journal of the Institute of Actuaries*, Vol. XXIV., p. 107, where the formula is derived from Gauss's theorem. It has been pointed out by the referee that this is a particular case of formula (42) of Mr. Sheppard's paper (*Proceedings*, Vol. XXXII., p. 270); but with this exception the formulæ given on pp. 269–70 appear to be distinct from those of this paper.

while the mean of (xi.) and (xii.) gives

$$A = \frac{h}{240}\left[396A_1 - 207A_2 + 52A_3 - A_6\right]$$

$$= \frac{h}{40}\left[11\,(u_0+u_6) + 66\,(u_1+u_5) - 3\,(u_2+u_4) + 92u_3\right], \qquad \text{(xvi.)}$$

with an error $h\left[\frac{1}{336}\delta^5 - \frac{583}{100800}\delta^7 + \dots\right]\mu\,(u_6 - u_0)$.

Again, if to (iv.) we add twice (vii.) and take the mean, we get

$$A = \frac{h}{7}\left[11A_1 - 5A_2 + A_3\right]$$

$$= \frac{h}{7}\left[2\,(u_0+u_6) + 11\,(u_1+u_5) + u_2 + u_4 + 14u_3\right], \qquad \text{(xvii.)}$$

with an error $\frac{h}{420}\left[\delta^3 - \frac{91}{84}\delta^5 + \frac{3017}{5040}\delta^7 - \dots\right]\mu\,(u_6 - u_0)$.

If we double (viii.) and add it to (xvi.), we get

$$A = \frac{h}{720}\left[1116A_1 - 495A_2 + 100A_3 - A_6\right]$$

$$= \frac{h}{120}\left[35\,(u_0+u_6) + 186\,(u_1+u_5) + 21\,(u_2+u_4) + 236u_3\right], \qquad \text{(xviii.)}$$

with an error $\frac{h}{5040}\left[\delta^5 - \frac{419}{60}\delta^7 + \dots\right]\mu\,(u_6 - u_0)$,

while the result of doubling (viii.) and adding it to (xiii.) is

$$A = \frac{h}{900}\left[1386A_1 - 603A_2 + 118A_3 - A_6\right]$$

$$= \frac{h}{150}\left[44\,(u_0+u_6) + 231\,(u_1+u_5) + 30\,(u_2+u_4) + 290u_3\right], \qquad \text{(xix.)}$$

with an error $-\frac{h}{12600}\left[\delta^5 + \frac{143}{12}\delta^7 + \dots\right]\mu\,(u_6 - u_0)$.

Formulæ of this kind can easily be extended.

6. By giving to n other values such as 8, 9, 10, ..., we get other groups; but in all these formulæ the earlier differences are got rid

of at the expense of increasing the coefficients of the later ones, and the larger the factors of n the greater is the increase in these coefficients. Thus, taking three factors a, b, c and p, q, r to satisfy the equations

$$pa^2+qb^2+rc^2 = 0,$$

$$pa^4+qb^4+rc^4 = 0,$$

we have

$$\frac{pa^6+qb^6+rc^6}{p+q+r} = -a^2b^2c^2, \quad \frac{pa^8+qb^8+rc^8}{p+q+r} = -a^2b^2c^2(a^2+b^2+c^2),$$

$$\frac{pa^{10}+qb^{10}+rc^{10}}{p+q+r} = -a^2b^2c^2(a^4+b^4+c^4+b^2c^2+c^2a^2+a^2b^2),$$

and so on; and these are elements of the coefficients in the expression for the error. The success of many of the formulæ involving six intervals appears to be due to some extent to the fact that *six* has as factors the first three natural numbers. If we put $n = 12$, we should get all the preceding formulæ duplicated, and a large number of others due to the introduction of the other factors. The degree of approximation is increased, but in practical applications the calculation of the ordinates often involves considerable numerical work, and it is desirable to combine a good degree of approximation with facility of computation. It is well known too that these differences run with great irregularity; they often change sign, and after first decreasing numerically they often increase rapidly in proceeding to the higher orders; so that a formula which is true to third differences only may give a better result than one which is true to the fifth or higher orders. A preliminary examination of the differences may guide us as to which set of formulæ is the best to use. This point is illustrated by the numerical examples given at the end of this paper.

7. If we take the ordinary central difference interpolation formula

$$u_x = u_0+\frac{x}{1!}\mu\delta u_0+\frac{x^2}{2!}\delta^2 u_0+\frac{x(x^2-1)}{3!}\mu\delta^3 u_0+\frac{x^2(x^2-1)}{4!}\delta^4 u_0+\ldots, \quad (4)$$

and integrate with respect to x, between limits $-\frac{1}{2}$ and $+\frac{1}{2}$, we get

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} u_x\,dx = \left[1+\tfrac{1}{24}\delta^2-\tfrac{17}{5760}\delta^4+\tfrac{367}{967680}\delta^6-\ldots\right]u_0$$

or
$$\int_0^h u_x\,dx = h\left[1+\tfrac{1}{24}\delta^2-\tfrac{17}{5760}\delta^4+\tfrac{367}{967680}\delta^6-\ldots\right]u_{\frac{1}{2}h}.$$

Hence
$$\int_0^{nh} u_x dx = h\left[1+E^h+E^{2h}+\ldots+E^{(n-1)h}\right]\int_0^h u_x dx$$
$$= h\left[u_{\frac{1}{2}h}+u_{\frac{3}{2}h}+\ldots+u_{(n-\frac{1}{2})h}\right]$$
$$+h\left[\lambda_2'\delta+\lambda_4'\delta^3+\lambda_6'\delta^5+\ldots\right](u_{nh}-u_0), \tag{5}$$

where λ_2', λ_4', ... stand for the numerical coefficients.

This corresponds to the mid-ordinate formula* given by Mr. Sheppard; but, as pointed out by him, it is not adapted for finding more accurate formulæ. Proceeding as before, we should get

$$\int_0^{nh} u_x dx = ah\left[u_{\frac{1}{2}ah}+u_{\frac{3}{2}ah}+\ldots+u_{(n-\frac{1}{2}a)h}\right]$$
$$+\frac{ah}{144}\left[6\delta+\left(\frac{23a^2}{40}-1\right)\delta^3-\left(\frac{145a^4}{1344}+\frac{23a^2}{80}-\frac{9}{20}\right)\delta^5-\ldots\right](u_{nh}-u_0), \tag{6}$$

where a is one of the factors of n.

The numerical coefficients here are greater than those of equation (3), and, as $a, b, c, \ldots$ must be odd numbers, the coefficients of the differences in the expression for the error will be much larger. It may be noticed, however, that the coefficients in (5) are smaller than the corresponding coefficients in (2); so that, if greater accuracy be required, it will probably be better to compute the first few differences. This, of course, involves a knowledge of terms preceding $u_{\frac{1}{2}h}$, and following $u_{(n-\frac{1}{2})h}$.

8. As illustrations of the preceding formulæ the values of the integrals
$$\int_0^1 \frac{dx}{1+x} = \log_e 2 \tag{a}$$
and
$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} \tag{b}$$
have been computed, using six intervals, with ordinates u_0, $u_{\frac{1}{6}}$, ..., u_1. It may be noticed that the central differences of $1/(1+x)$ first decrease numerically, then increase and become infinite, since $u_{-1}=\infty$; so that after a certain point the expressions obtained above cease to represent the error. For u_0 the odd differences begin to increase with the seventh; for u_1 the increase begins much later. In the case of $1/(1+x^2)$ the odd central differences of u_0 are all zero, and those of u_1 begin to increase numerically after the third.

The true values of (a) and (b) to seven places of decimals are ·6931472, and ·7853982.

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* *Proceedings*, Vol. XXXII., p. 267.

The values of the ordinates are

x	$u = 1/(1+x)$	$u = 1/(1+x^2)$
0	1·	1·
$\frac{1}{6}$	·8571429	·9729730
$\frac{2}{6}$	·75	·9
$\frac{3}{6}$	·6666667	·8
$\frac{4}{6}$	·6	·6923077
$\frac{5}{6}$	·5454545	·5901639
1	·5	·5

and the errors in the computed values are as follows:—

Formula.	No. of terms in formula.	Error in (a) $\times 10^7$.	Error in (b) $\times 10^7$.
(i.) (Simpson's rule)	7	+ 226	− 3
(ii.) (Simpson's second rule)	7	+ 482	− 23
(vi.)	5	−146	+ 28
(viii.) (Weddle's rule)	7	+ 22	+ 14
(xiii.) (Hardy's formula)	5	− 15	−179
(xiv.)	7	+ 9	− 55
(xv.)	7	+ 40	+ 12
(xvii.)	7	− 66	+ 21
(xviii.)	7	+ 7	− 67
(xix.)	7	+ 10	− 50

It appears that, while any one of these formulæ will give a good approximation, the best results are not obtained by always using the same ones. When once the values of the ordinates have been obtained, that of the integral can be readily computed by several of

these formulæ; and, as some err in excess and others in defect, we shall get very close limits within which the true value lies, and generally better results than would be attained by exclusive use of any one formula.

Thursday, April 10th, 1902.

Dr. HOBSON, F.R.S., President, in the Chair.

Eleven members present.

Prof. C. J. Joly, M.A., Dunsink Observatory, Ireland; Ganesh Prasad, D.Sc., Christ's College, Cambridge, and Miss Lilian Janie Whitley, B.A., Westfield College, Hampstead, N.W., were elected members.

The President (Dr. Larmor temporarily in the Chair) communicated a "Note on Divergent Series." Prof. Love next gave results he had arrived at in connection with "Stress and Strain in two-dimensional Elastic Systems." Discussions followed on both communications, in which the President and Messrs. Larmor and Love took part.

The President read the titles of the following papers:—

Further applications of Matrix Notation to Integration Problems: Dr. H. F. Baker.

On the Convergence of Series which represent a Potential: Prof. T. J. I'A. Bromwich.

On the Groups defined for an Arbitrary Field by the Multiplication Tables of certain Finite Groups: Dr. L. E. Dickson.

The following presents were made to the Library:—

"Educational Times," April, 1902.

"Indian Engineering," Vol. xxxi., March 15–April 5, 1902.

Gibbs, J. Willard.—"Elementary Principles in Statistical Mechanics," 8vo; London, 1902.

"Nautical Almanac for 1905," 8vo; Edinburgh, 1902.

"Mittheilungen der Mathematischen Gesellschaft," Bd. iv., Heft 2; Hamburg, 1902.

"Supplemento al Periodico di Matematica," Anno v., Fasc. 5; Livorno, 1902.

"Il Pitagora," Anno viii., Nos. 1-5; Palermo, 1901-2.

"Memoirs of the National Academy of Sciences," Vol. viii.; Washington, 1898.

Dickson, Dr. L. E.—"College Algebra"; New York, 1902.

From the "Scientia" Series, presented by the publisher, M. C. Naud:—

"Cryoscopie," par F. M. Raoult, No. 13.

"Franges d'Interférences et leurs applications métrologiques," par J. Macé de Lépinaz, No. 14.

"La Géométrie non-euclidienne," par P. Barbarin, No. 15.

"Le Phénomène de Kerr," par E. Néculcéa, No. 16.

"Théorie de la Lune," par H. Andoyer, No. 17.

"Géométrographie," par E. Lemoine, No. 18.

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. lxix., Nos. 457, 458; 1902.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxvi., Heft 4; Leipzig, 1902.

"Rendiconti del Circolo Matematico di Palermo," Tomo xvi., Fasc. 1, 2; 1902.

"Bulletin of the American Mathematical Society," Vol. viii., Nos. 5, 6; New York, 1902.

"Bulletin des Sciences Mathématiques," Tome xxvi., Fév., 1902; Paris.

"Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche," Vol. viii., Fasc. 2; Napoli, 1902.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. xi., Fasc. 5; Roma, 1902.

"Jahresbericht der Deutschen Mathematiker-Vereinigung in Monatsheften," herausgegeben von A. Gutzmer in Jena, Band xi., 1 and 2 (doppel-) Heft (Jan.-Feb.), Dec. 19, 1901; Band xi., 3 Heft (März), Feb., 1902.

Further Applications of Matrix Notation to Integration Problems.

By H. F. BAKER. Received and communicated April 10th, 1902.

CONTENTS.

1. Gives the finite equations of the adjoint group of a continuous group with parameters not canonical in terms of "the matrix ξ" of the first and second parameter groups.

2. Gives the finite equations of the adjoint group with canonical parameters, showing that the exponential matrix Δ of a former note (*Proc. Lond. Math. Soc.*, Vol. XXXIV., 1902, p. 91) enters therein; and so gives a simple proof of the so-called exponential theorem.

3. Remarks on the connection with Lie's formulæ.

4. Shows that any transformation of the adjoint group can be resolved into a succession of two transformations respectively of the first and second parameter groups.

5. Shows that this leads to a result including as a particular case the theorem that the characteristic determinantal equation allows the adjoint group.

6. Gives another proof of this theorem of invariance.

7. Develops the result further, establishing in particular the equation

$$\psi(e't, \Delta_{e'}x) = \Delta_{e'(1-t)}\psi(x, e't).$$

8. Remarks on the translation from the first to the second parameter group.

9. Obtains a proof of the existence of and a formula for the general integral of any set of simultaneous linear differential equations with variable coefficients, *in a form valid for the whole of the Mittag-Leffler star region over which the integrals exist.* This appears a remarkable result.

10. Remarks that the theorem and method of § 9 is particularly applicable to establish the existence of and to calculate the integrals for a system of differential equations with *real* independent variable, the coefficients not being necessarily continuous, so long as they are integrable.

11, 12. Remark on the generalization of the results of §§ 9, 10.

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1. The finite equations of a continuous group of n variables $x_1, \ldots, x_n$ and r parameters $a_1, \ldots, a_r$, being

$$x_i = f_i(x^0, a),$$

leading to equations

$$f_i\left[f(x^0, a), b\right] = f_i\left[x^0, \phi(a, b)\right],$$

so that $\qquad y_\sigma = \phi_\sigma(y^0, a) \quad \text{and} \quad z_\sigma = \phi_\sigma(a, z^0),$

for $\sigma = 1, \ldots, r$, are the finite equations respectively of the first and second parameter groups, we have the three sets of differential equations

$$\frac{\partial x}{\partial a} \alpha(a) = \xi(x), \quad \frac{\partial y}{\partial a} \alpha(a) = \alpha(y), \quad \frac{\partial z}{\partial a} \beta(a) = \beta(z),$$

where $\alpha(a)$ is a matrix of r rows and columns, and $\xi(x)$ a matrix of n rows and r columns. If, now, in the general infinitesimal transformation of the group

$$\sum_{\sigma=1}^{r} e_\sigma P_\sigma = \sum_{\sigma=1}^{r} e_\sigma \left(\xi_{1\sigma} \frac{\partial}{\partial x_1} + \ldots + \xi_{n\sigma} \frac{\partial}{\partial x_n} \right),$$

we change the independent variables to $x_1^0, \ldots, x_n^0$ by means of the equations of the group

$$x_i = f_i(x^0, a),$$

the infinitesimal transformation becomes $\sum_{\sigma=1}^{r} e_\sigma^0 P_\sigma^0$, where P_σ^0 contains $x_1^0, \ldots, x_n^0$, just as P_σ contains $x_1, \ldots, x_n$ and $e_1, \ldots, e_r$ are linearly deduced from $e_1^0, \ldots, e_r^0$ by equations, depending on $a_1, \ldots, a_r$ only, which for varying $a_1, \ldots, a_r$ are the finite equations of the adjoint group.

These equations are, in fact,

$$e = \alpha^{-1} \beta . e^0,$$

namely,*

$$e_\sigma = \sum_{\tau=1}^{r} \sum_{\rho=1}^{r} (\alpha^{-1})_{\sigma\rho} \beta_{\rho\tau} e_\tau^0 .$$

2. The infinitesimal equations of the adjoint group are, however,

$$-\left(E_{1\sigma} \frac{\partial}{\partial e_1} + \ldots + E_{r\sigma} \frac{\partial}{\partial e_r} \right),$$

where

$$E_{\rho\sigma} = \sum_{\tau=1}^{r} c_{\sigma\tau\rho} e_\tau,$$

and the finite equations of the adjoint group, with *canonical parameters* $e_1', \ldots, e_r'$, are therefore to be obtained by solving the r ordinary differential equations

$$\frac{de}{dt} = -Ee' = E'e,$$

* The notation in the *Encyk. Math. Wiss.*, Vol. II., A. 6, p. 406, seems an unfortunate change from Lie's for the matrix α in *Transformationsgruppen*, Vol. I., p. 34. If the quantity $a_{\rho\sigma}$ of the *Encyk.* be called $\gamma_{\sigma\rho}$, then the matrix γ is the negative of the matrix $\beta(a)$ belonging to the second parameter group.

making $e = e^0$, when $t = 0$, and afterwards putting $t = 1$. These equations are, however, obviously satisfied* by

$$e = \left(1 + tE' + \frac{t^2}{2!}E'^2 + \frac{t^3}{3!}E'^3 + \ldots\right) e^0.$$

Thus, if $$\Delta' = 1 + E' + \frac{E'^2}{2!} + \frac{E'}{3!} + \ldots,$$

the finite equations of the adjoint group are $e = \Delta' e^0$, the parameters being canonical. Using these same parameters to write the finite equations of the original group in the form

$$x_i = F_i\,(x^0, e'),$$

we have the result that the equations

$$x_i = F_i\,(x^0, e'), \quad e = \Delta' e^0$$

lead to $$\sum_{\sigma=1}^{r} e_\sigma P_\sigma = \sum_{\sigma=1}^{r} e_\sigma^0 P_\sigma^0.$$

Considering then a further set

$$y_i = F_i\,(x, e''), \quad f = \Delta'' e, \quad \Sigma f_\sigma Q_\sigma = \Sigma e_\sigma P_\sigma,$$

we obtain the set

$$y_i = F_i\,(x^0, e'''), \quad f = \Delta''\Delta' e^0, \quad \Sigma f_\sigma Q_\sigma = \Sigma e_\sigma^0 P_\sigma^0,$$

showing that the equations of the parameter group, with canonical variables and parameters

$$e_\sigma''' = \psi_\sigma\,(e', e''),$$

lead to $$\Delta''' = \Delta''\Delta'.$$

Thus we have a very simple proof of the so-called exponential theorem.

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* A particular example of this integration is the proof of the formulæ for the nine cosines between two sets of rectangular axes by integrating the equations

$$\frac{dx}{dt} = ny - mz, \quad \frac{dy}{dt} = -nx + lz, \quad \frac{dz}{dt} = mx - ly.$$

The matrix $$E = \begin{pmatrix} 0 & n & -m \\ -n & 0 & l \\ m & -l & 0 \end{pmatrix}$$

satisfies the equation $$E^3 = -(l^2 + m^2 + n^2)\,E\,;$$

cf. Proc. Lond. Math. Soc., Vol. xxxiv., p. 116.

3. The problem of finding canonical variables for a group of which some form of the finite equations is known is thus, for a group without special infinitesimal transformations, that of solving the r^2 equations

$$\alpha^{-1}(a)\,\beta(a) = \Delta',$$

the matrices ρ and σ of Lie [*Transformationsgruppen*, Vol. III., p. 613, equation (24)] being these two respectively. And by expressing that the two parameter groups are reciprocal, or otherwise [Lie, *Transformationsgruppen*, Vol. III., p. 616, equation (27)], it can be shown that

$$(e_1'A_1 + \ldots + e_r'A_r)(\alpha^{-1}\beta) = E'(\alpha^{-1}\beta),$$

where

$$A_\sigma = \sum_{\rho=1}^{r} a_{\rho\sigma}\frac{\partial}{\partial a_\rho}$$

is the general infinitesimal transformation of the first parameter group, while the exponential theorem in general variables has a form

$$\alpha_2^{-1}\beta_2 = \alpha_1^{-1}\beta_1 \,.\, \alpha^{-1}\beta.$$

4. We prove now that any transformation $y = \Delta' x$ of the adjoint group can be resolved into a succession of two transformations

$$z_\sigma = \psi_\sigma(x, e'), \quad y_\sigma = \psi_\sigma(-e', z),$$

respectively, of the first and second parameter groups. Herein the variables x, y, z, as well as the parameters e', are canonical, a fact we emphasise by using the functional sign ψ instead of the more usual ϕ.

Using infinitesimal transformations, we have

$$\begin{aligned} y_\sigma &= z_\sigma - [e_1'\beta_{\sigma 1}(z) + \ldots + e_r'\beta_{\sigma r}(z)] + \ldots \\ &= x_\sigma + e_1'[\alpha_{\sigma 1}(x) - \beta_{\sigma 1}(x)] + \ldots + e_\sigma'[\alpha_{\sigma r}(x) - \beta_{\sigma r}(x)] + \ldots . \end{aligned}$$

On the other hand (*Proc. Lond. Math. Soc.*, Vol. XXXIV., 1902, p. 97), by Schur's formulæ,

$$\alpha - \beta = \alpha - \alpha\Delta = -E.$$

Thus the infinitesimal transformations of the compound transformation are these of the adjoint group.

5. This remark is equivalent with

$$\Delta' x = \psi[-e', \psi(x, e')],$$

namely, as

$$\psi(e, e') = -\psi(-e', -e),$$

with

$$-\Delta' x = \psi[-\psi(x, e'), e'],$$

and gives $\quad -\psi(x, e') = \psi(-\Delta' x, -e') = -\psi(e', \Delta' x)$;

and hence $\quad \psi_\sigma(e', \Delta' x) = \psi_\sigma(x, e')$,

a result* leading, by the exponential theorem, to

$$\Delta_y \Delta_{e'} = \Delta_{e'} \Delta_x,$$

as a consequence solely of the linear equations

$$y = \Delta' x.$$

As these last are unaffected by multiplying each of $x_1, \ldots, x_r$; $y_1, \ldots, y_r$ by the arbitrary numerical quantity λ, we can, by comparing coefficients of λ in the equation

$$\Delta_{\lambda y} \Delta_{e'} = \Delta_{e'} \Delta_{\lambda x},$$

infer that $\quad E_y \Delta_{e'} = \Delta_{e'} E_x$;

and hence, if θ be an arbitrary numerical quantity, as the determinant of $\Delta_{e'}$ is not zero, we see that

$$| E_x + \theta | = | E_y + \theta |,$$

namely, that the so-called characteristic equation expressed by the vanishing of the determinant $| E_x + \theta |$ allows the transformations of the adjoint group. The equation

$$E_y + \theta = \Delta_{e'} (E_x + \theta) \Delta_{-e'}$$

shows that the greatest common divisor, in θ, of the first minors of $| E_x + \theta |$ agrees with that for $| E_y + \theta |$, and hence that, if

$$\chi(E) = 0$$

be the algebraic equation satisfied by the matrix E, the equation

$$\chi(\omega) = 0,$$

for arbitrary ω, allows the adjoint group. Further, the determinantal equation

$$| \Delta_x - e^{-\theta} | = 0,$$

which is satisfied by all the roots θ of the characteristic equation, also allows the adjoint group; this is equivalent with the fact that the determinantal equation

$$| \alpha(a) - \lambda \beta(a) | = 0,$$

for arbitrary numerical λ, allows the adjoint group; and *herein the variables $a_1, \ldots, a_r$ need not be canonical.*

---

* Re-proving that, if T_e be the transformation of the original group with canonical parameters e, the canonical parameters of $T_f = T_{e'} T_e T_{e'}^{-1}$ are $f = \Delta' e$; or conversely.

6. In the general case when the group has no special infinitesimal transformations the equation

$$\Delta_y \Delta_{e'} = \Delta_{e'} \Delta_x$$

necessarily involves the equations

$$\psi_s(e', y) = \psi_s(x, e'),$$

as was proved in my previous note (*Proc. Lond. Math. Soc.*, Vol. XXXIV., 1902, p. 112); so that we have the identities

$$\psi_s(e', \Delta' x) = \psi_s(x, e'),$$

which therefore hold also in the special case. We give now a direct proof of the equation

$$\Delta_y \Delta_{e'} = \Delta_{e'} \Delta_x,$$

where

$$y = \Delta' x$$

(*cf.* Engel, *Math. Annal.*, Vol. XXXI., 1882, p. 262, footnote).

If $e^1 = E'e$, and e'' be a further independent set, we have (*Proc. Lond. Math. Soc.*, Vol. XXXIV., p. 93)

$$E^1 e'' = -E'' e^1 = -E''E'e = EE''e' + E'Ee'' = E'Ee'' - EE'e'':$$

so that

$$E^1 = E'E - EE'.$$

Put, similarly,

$$e^2 = E'e^1 = E'^2 e, \quad e^3 = E'e^2 = E'^3 e,$$

and so on; then

$$\Delta' e = e + E'e + \frac{E'^2 e}{2!} + \frac{E'^3 e}{3!} + \dots$$

$$= e + e^1 + \frac{1}{2!} e^2 + \frac{1}{3!} e^3 + \dots;$$

and therefore, if $f = \Delta' e$, as E_f is linear in f,

$$E_f = E + E^1 + \frac{1}{2!} E^2 + \frac{1}{3!} E^3 + \dots$$

$$= E + (E'E - EE') + \frac{1}{2!}(E'E^1 - E^1 E') + \frac{1}{3!}(E'E^2 - E^2 E') + \dots$$

$$= E + (E'E - EE') + \frac{1}{2!}(E'^2 E - 2E'EE' + EE'^2)$$

$$+ \frac{1}{3!}(E'^3 E - 3E'^2 EE' + 3E'EE'^2 - EE'^3) + \dots$$

$$= \left(1+E'+\frac{E'^2}{2!}+\dots\right) E-\left(1+E'+\frac{E'^2}{2!}+\dots\right) EE'$$

$$+\frac{1}{2!}\left(1+E'+\frac{E'^2}{2!}+\dots\right) EE'^2-\dots$$

$$= \Delta' E\,(\Delta')^{-1},$$

giving $$E_f\Delta' = \Delta' E$$

as a consequence of $$f = \Delta' e.$$

From this $$E_f^k\Delta' = \Delta' E^k \quad\text{and}\quad \Delta_f\Delta' = \Delta'\Delta_e.$$

7. Hence it follows that the two sets of differential equations

$$\frac{d\eta}{dt} = \beta(\eta)\,e', \quad \frac{d\xi}{dt} = \alpha(\xi)\,e'$$

are transformable into one another by the linear equations

$$\eta = \Delta_{e'(1-t)}\,\xi.$$

And this leads to $\psi(e't,\ \Delta_{e'}x) = \Delta_{e'(1-t)}\psi(x,\ e't)$,
including the former result.

For, from $\eta = \Delta_{e'(1-t)}\,\xi = \Delta_{-e't}\Delta_{e'}\,\xi$, $\dfrac{d\xi}{dt} = \alpha(\xi)\,e'$,

we have $$\frac{d\eta}{dt} = -E'\Delta_{-e't}\Delta_{e'}\xi+\Delta_{-e't}\Delta_{e'}\frac{d\xi}{dt}$$

$$= -E'\eta+\Delta_{e'(1-t)}\alpha(\xi)\,e',$$

while, by Schur's formulæ, from

$$E_\eta\Delta_{e'(1-t)} = \Delta_{e'(1-t)}E_\xi,$$

we have (*Proc. Lond. Math. Soc.*, Vol. XXXIV., p. 96)

$$\alpha(\eta)\,\Delta_{e'(1-t)} = \Delta_{e'(1-t)}\,\alpha(\xi),$$

and hence $$\frac{d\eta}{dt} = -E'\eta+\alpha(\eta)\,\Delta_{e'(1-t)}e'$$

$$= E_\eta e'+\alpha(\eta)\,e'$$

$$= [E_\eta+\alpha(\eta)]\,e'$$

$$= \beta(\eta)\,e',$$

since $$\alpha-\beta = -E$$

(*ib.*, p. 97). This proves the transformation in question.

8. It should perhaps be remarked that, if in

$$x = \psi(x^0, e')$$

we put
$$x^0 = \psi(e^0, -y^0),$$

we obtain

$$x = \psi\left[e^0, \psi(-y^0, e')\right] = \psi\left[e^0, -\psi(-e', y^0)\right] = \psi(e^0, -y),$$

provided
$$y = \psi(-e', y^0).$$

These equations give the translation of the first parameter group into the second.

9. The method used above for obtaining the finite equations of the adjoint group can be generalized as follows to obtain an expression for the integrals of any set of simultaneous linear equations *with variable coefficients.*

Let the equations be

$$\frac{dx_i}{dt} = u_{i1}x_1 + \ldots + u_{in}x_n \quad (i = 1, \ldots, n),$$

and let $t = t^0$, which, by a change of independent variable, we shall suppose to be $t = 0$, be a value of t about which every one of the n^2 coefficients u_{ij} is developable, and therefore single-valued, finite, and continuous. By barriers passing to infinity from the infinities or branch points of these n^2 coefficients, we define a simply-connected region of the plane over the whole of the finite portion of which this character remains; and we suppose t limited to this finite region.

Now, writing the equations in the form

$$\frac{dx}{dt} = ux,$$

form in succession the matrices

$$u^{(1)} = \int_0^t u\,dt, \quad u^{(2)} = \int_0^t uu^{(1)}dt, \quad \ldots, \quad u^{(n+1)} = \int_0^t uu^{(n)}dt, \quad \ldots,$$

where, by the integral of a matrix is meant simply the matrix whose elements are the integrals of the elements of the original; if the operation of integrating a matrix from 0 to t be denoted by the prefix Q, these successive matrices may be succinctly denoted by

$$u^{(1)} = Qu, \quad u^{(2)} = QuQu, \quad u^{(3)} = QuQuQu, \quad \ldots,$$

it being understood that each symbol Q operates on all that follows it.

It is now to be proved that the elements of the matrix

$$\nabla = 1 + Qu + Qu\,Qu + Qu\,Qu\,Qu + \dots \text{ to infinity}$$

are all uniformly and absolutely converging over the whole of the star region before described, and that the values

$$x = \nabla x^0$$

represent the solutions of the original differential equations which reduce to the arbitrary values $x_1^0, \dots, x_n^0$ for $t = 0$.

Let M_{ij} be a real positive quantity not exceeded by the modulus of u_{ij} for any value of t within the region under consideration; every one of the functions

$$u_{ij}^{(1)}(t) = \int_0^t u_{ij}(t)\,dt,$$

$$u_{ij}^{(2)}(t) = \int_0^t \left[u_{i1}(t)\,u_{1j}^{(1)}(t) + \dots + u_{in}(t)\,u_{nj}^{(1)}(t)\right] dt,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

is developable, single-valued, finite, and continuous about every point throughout the star region under consideration; let t_1 be a particular value of t on the path of integration from 0 to t, and s_1 the length of the path of integration to this point, the whole arc from 0 to t being s; then

$$|u_{ij}^{(1)}(t)| \overline{\overline{<}}\, M_{ij}\int_0^s ds_1 \overline{\overline{<}}\, sM_{ij},$$

and, in the same way,

$$|u_{ij}^{(1)}(t_1)| \overline{\overline{<}}\, s_1 M_{ij};$$

also
$$|u_{ij}^{(2)}(t)| \overline{\overline{<}} \int_0^s s_1\,ds_1\,(M_{i1}M_{1j} + M_{i2}M_{2j} + \dots + M_{in}M_{nj})$$

$$\overline{\overline{<}}\, (M^2)_{ij}\int_0^s s_1\,ds_1$$

$$\overline{\overline{<}}\, \tfrac{1}{2}s^2\,(M^2)_{ij},$$

where M denotes the matrix of the quantities M_{ij}, and M^2 the square of this matrix; in the same way,

$$|u_{ij}^{(2)}(t_1)| \overline{\overline{<}}\, \tfrac{1}{2}s_1^2\,(M^2)_{ij};$$

also
$$|u_{ij}^{(3)}(t)| \overline{\overline{<}} \int_0^s \tfrac{1}{2}s_1^2\,ds_1\left[M_{i1}(M^2)_{1j} + \dots + M_{in}(M^2)_{nj}\right]$$

$$\overline{\overline{<}}\, \frac{s^3}{3!}(M^3)_{ij};$$

this process can be continued indefinitely. It follows that each of the n^2 infinite series constituting the elements of the matrix

$$\nabla = 1 + u^{(1)} + u^{(2)} + u^{(3)} + \dots$$

has terms whose moduli are respectively equal to or less than the real positive terms of the corresponding infinite series constituting the elements of the matrix

$$1 + sM + \frac{s^2}{2!}M^2 + \frac{s^3}{3!}M^3 + \dots .$$

This last is, however, certainly convergent for all finite values of s; its sum is, in fact, given in the case in which the algebraic equation satisfied by the matrix M has unequal roots by the formula given, *Proc. Lond. Math. Soc.*, Vol. XXXIV., p. 114, which can be easily modified to suit the case of repeated roots. Each of the elements of the matrix ∇ is thus an absolutely and uniformly converging series for the whole of the star region considered, and represents an analytic function of t, single-valued, finite, continuous, and developable throughout; and differentiation and integration term by term of the series are permissible.

Hence, if we take

$$x = (1 + Qu + Qu\,Qu + \dots)\, x^0,$$

we obtain
$$\frac{dx}{dt} = u\,(1 + Qu + Qu\,Qu + \dots)\, x^0 = ux,$$

showing that $x = \nabla x^0$ satisfies the differential equations and reduces to x_0 when $t = 0$.*

---

* If $\omega_1, \dots, \omega_r$ be functions of t, and $A_1, \dots, A_r$ the infinitesimal transformations of the first parameter group (*cf.* §§ 1, 2), we have the important application that solutions of the partial equation

$$\frac{\partial F}{\partial t} + \sum_{\sigma=1}^{r} \omega_\sigma A_\sigma F = 0$$

are given by $\beta^{-1}ae$, where $e_1, \dots, e_r$ satisfy the linear equations

$$\frac{de}{dt} = -E\omega.$$

The formula of the text leads, for the non-homogeneous equation

$$\frac{dx}{dt} = ux + c,$$

to the solutions
$$x = \nabla x_0 + \nabla Q \nabla^{-1} c\,;$$

thus the solutions of the equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{n} (c_i + u_{i1}x_1 + \dots + u_{in}x_n)\frac{\partial f}{\partial x_i} = 0,$$

reducing to x when $t = 0$, are $\nabla^{-1}x - Q\nabla^{-1}c$.

Example (i.)—For an ordinary differential equation with one independent and one dependent variable

$$\frac{dx}{dt} = ux,$$

we easily verify that

$$Qu\,Qu = \frac{1}{2!}(Qu)^2, \quad Qu\,Qu\,Qu = \frac{1}{3!}(Qu)^3, \text{ \&c.},$$

and so get the ordinary solution

$$x = (\text{exp. } Qu)\, x^0.$$

Example (ii.) — The ordinary linear differential equation of the second order

$$\frac{d^2x}{dt^2} = wx$$

is equivalent to the pair $dx/dt = x'$, $dx'/dt = wx$,

namely, to
$$\left(\frac{dx}{dt}, \frac{dx'}{dt}\right) = \begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix} (x, x').$$

Putting then
$$u = \begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix},$$

we find
$$x = \Delta_1 x_0 + \Delta_2 x_0',$$

where Δ_1, Δ_2 are the series

$$\Delta_1 = 1 + Q^2w + Q^2w\,Q^2w + Q^2w\,Q^2w\,Q^2w + \dots,$$

$$\Delta_2 = t + Q^2wt + Q^2w\,Q^2wt + Q^2w\,Q^2w\,Q^2wt + \dots,$$

as the solution for which x and dx/dt have the values x_0 and x_0' for $t = 0$, it being understood that Q^2w denotes $Q\,(Qw)$, and so on, and Q^2wt denotes $Q\left[Q\,(wt)\right]$, and so on. It is manifest that the terms Q^2w, Q^2wt, Q^2wQ^2w, Q^2wQ^2wt, ... vanish respectively to the orders 2, 3, 4, 5, ..., when $t = 0$, and, if w be a linear aggregate $\lambda_1 w_1 + \lambda_2 w_2 + \dots$ of other functions $w_1, w_2, \dots$ with constant coefficients, $\lambda_1, \lambda_2, \dots$, they are homogeneous polynomials in $\lambda_1, \lambda_2, \dots$, respectively, of dimensions 1, 1, 2, 2, ...

For instance, if in Bessel's equation

$$\xi^2 d^2x/d\xi^2 + \xi\, dx/d\xi + (\xi^2 - n^2)\, x = 0$$

we put, with arbitrary a,

$$\xi = ae^{\frac{1}{2}t}, \quad m = \tfrac{1}{4}n^2, \quad c = \tfrac{1}{4}a,$$

we obtain $\qquad d^2x/dt^2 = (m-ce^t)\,x \quad \text{or} \quad w = m-ce^t.$

Hence we find

$$\Delta_1 = 1+\left\{m\frac{t^2}{2!}-c\,(e^t-1-t)\right\}$$
$$+\left\{m^2\frac{t^4}{4!}+mc\left[4+2t+\frac{t^2}{2!}+\frac{t^3}{3!}-e^t\,(4-2t+\tfrac{1}{2}t^2)\right]\right.$$
$$\left.+c^2\left[-\tfrac{5}{4}-\tfrac{1}{2}t+e^t\,(1-t)+\tfrac{1}{4}e^{2t}\right]\right\}+\dots,$$

$$\Delta_2 = t+\left\{m\frac{t^3}{3!}-c\left[(t-2)\,e^t+t+2\right]\right\}$$
$$+\left\{m^2\frac{t^2}{5!}+mc\left[-8-4t-t^2-\frac{t^3}{3!}+e^t\left(8-4t+t^2-\frac{t^3}{3!}\right)\right]\right.$$
$$\left.+c^2\left[\tfrac{3}{4}+\tfrac{1}{4}t+te^t+\tfrac{1}{4}\,(t-3)\,e^{2t}\right]\right\}+\dots.$$

Since these are convergent for all finite values of t, they can be rearranged in powers of t, and will then agree with the power series found in the ordinary way.

If, on the other hand, we take Legendre's equation

$$(t^2-1)\,d^2y/dt^2+2t\,dy/dt = n\,(n+1)\,y,$$

and put $\qquad y\sqrt{t^2-1} = x,$

it reduces to $\qquad d^2y/dt^2 = wx,$

with $\qquad w = n\,(n+1)(t^2-1)^{-1}-(t^2-1)^{-2}.$

Setting aside the practical difficulty of expressing the successive quadratures of w in explicit form, the series obtained by the theorem is valid for the whole finite plane of t with the exception of the two barriers joining the points $t=-1$, $t=1$ to infinity. But when rearranged in powers of t it ceases to converge outside the circle of radius unity whose centre is at $t=0$. In other words, the theorem may be looked upon as giving a rearrangement of the ordinary power series solutions, so that they continue to converge over the whole star region described.

10. In the previous section we have for simplicity dealt with the case of analytical functions of complex variables. But it is clear that if in the n linear equations

$$\frac{dx}{dt} = ux$$

the elements u_{ij} be functions of the real variable t, which between $t = 0$ and $t = a$ are single valued, throughout finite, and integrable, the method establishes the existence of integrals of the equations within these limits, and gives a rule for calculating them, which, for instance, with a mechanical integrator may be of practical utility for purposes of computation.

11. If the roots θ_1, θ_2, ... of the algebraic equation satisfied by a matrix M, supposed unequal, be within the circle of convergence of the series

$$f(x) = 1 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \ldots,$$

it is immediately proved that the series

$$1 + a_1\frac{M}{1!} + a_2\frac{M^2}{2!} + a_3\frac{M^3}{3!} + \ldots$$

is convergent, its sum being given by the formula of *Proc. Lond. Math. Soc.*, Vol. XXXIV., 1902, p. 114, by replacing therein e^{θ_1}, e^{θ_2}, ..., respectively by $f(\theta_1)$, $f(\theta_2)$, ... The case of equal roots follows at once from this formula.

If, now, u_1, u_2, ... be matrices of the same order, such that for a certain range of values of the independent variable t their elements are less in absolute value than those of a matrix M which are real and positive, if c_1, c_2, ... be numerical quantities whose moduli are a_1, a_2, ..., it is easily proved, as in § 9, that the convergence of the series

$$\nabla = 1 + c_1 Qu_1 + c_2 Qu_1 Qu_2 + c_3 Qu_1 Qu_2 Qu_3 + \ldots$$

is involved in that of

$$1 + a_1 M + \frac{a_2 M^2}{2!} + \frac{a_3 M^3}{3!} + \ldots.$$

In particular, if $f(x)$ be an integral function, the series ∇ converges in a star region excluding only the singularities of the elements of the matrices u_1, u_2, ...

12. The discussion of the particular case when $u_1, u_2, \ldots$ are single finite integrable functions of t is scarcely cognate to the present note, but seems worth remark. Suppose that, for instance,

$$u_r = 1 + u_1^{(r)} t + u_2^{(r)} t^2 + \ldots$$

are all polynomials in t, reducing to unity when $t = 0$; then the series

$$\nabla = 1 + c_1 Q u_1 + c_2 Q u_1 Q u_2 + \ldots$$

is a series of polynomials converging within a star figure excluding the singularities of $u_1, u_2, \ldots$, and lying within the circle of convergence of the series

$$1 + a_1 M t + \frac{a_2}{2!} M^2 t^2 + \ldots,$$

where $$a_r = |c_r| \quad \text{and} \quad |u_r| < M;$$

and, if $$\nabla = F(t),$$

we have $$c_r = \left[\frac{1}{u_r}\frac{d}{dt}\frac{1}{u_{r-1}}\frac{d}{dt} \cdots \frac{1}{u_1}\frac{d}{dt}F(t)\right]_{t=0}.$$

When each of $u_1, u_2, u_3, \ldots$ reduces to unity, we have Taylor's theorem; the same also, after a change of variable, when

$$u_1 = u_2 = \ldots .$$

In general, if it were possible to choose the polynomials $u_1, u_2, \ldots$, for a given function $F(t)$, so that the function

$$1 + c_1 t + \frac{c_2 t^2}{2!} + \ldots$$

was an integral function, we should have an expansion of $F(t)$ in polynomials valid over the whole of a finite star region excluding only the singularities of $F(t)$ and $u_1, u_2, \ldots$.

On Quantitative Substitutional Analysis (Second Paper). *By* A. Young. Received and read February 13th, 1902. Revised April 2nd, 1902.

In § 16 of my first paper on the above subject* a series of substitutional expressions was obtained, which was written

$$1 = \Sigma A_{a_1, a_2, \ldots, a_h} T_{a_1, a_2, \ldots, a_h}.$$

The object of the first section of the present paper is to find the coefficient $A_{a_1, a_2, \ldots, a_h}$.

The second section deals with the relations between the forms PN of which one of the terms T of the above series is made up.

The latter part of the paper is devoted to the application of the theory already developed to modern algebra. By means of the above series it is shown that every integral function of the coefficients of any q-ary quantics may be expressed linearly in terms of coefficients of concomitants of these quantics.

The fifth section is devoted to the invariants of a single binary n-ic; it is shown that these may be expressed in terms of forms $f(a_0, a_1, \ldots, a_n)$, where $a_0, a_1, \ldots, a_n$ are certain numbers which completely define the invariant, and are such that

$$\begin{aligned} a_0 + a_1 + \ldots + a_n &= \text{the degree,} \\ a_1 + 2a_2 + \ldots + na_n &= a_{n-1} + 2a_{n-2} + \ldots + na_0 \\ &= \text{the weight.} \end{aligned}$$

I. *The Coefficients* $A_{a_1, a_2, \ldots, a_h}$ *in the Substitutional Expansion* $1 = \Sigma A_{a_1, a_2, \ldots, a_h} T_{a_1, a_2, \ldots, a_h}$.

1. $T_{a_1, a_2, \ldots, a_h}$ is defined as follows:—

The letters $a_1, a_2, \ldots, a_n$ are arranged in any manner in h horizontal rows, so that each row has its first letter in the same vertical column, its second letter in a second vertical column, and

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* *Proc. Lond. Math. Soc.*, Vol. XXXIII., p. 97.

so on; there being a_1 letters in the first row, a_2 in the second, &c., and finally, a_h in the last; the a's satisfying the relations

$$a_1+a_2+\ldots+a_h = n, \quad a_1 \not< a_2 \not< a_3 \ldots \not< a_h.$$

From this table an expression

$$S = \Gamma_1'\Gamma_2' \ldots \Gamma_h' G_1 G_2 \ldots G_k$$

is formed, such that Γ_1' is the negative symmetric group of the letters of the first row, Γ_2' that of the letters of the second row, and so on; G_1 is the positive symmetric group of the letters of the first column, G_2 of the second column, and so on (it being understood that the positive or negative symmetric group of a single letter is unity). Then $T_{a_1, a_2, \ldots, a_h}$ is the sum of the $n!$ expressions—not necessarily all different—obtained by permuting the letters in S in all possible ways.*

The product of the negative symmetric groups in S will be denoted by N, and that of the positive symmetric groups by P. It was proved that

$$T = \Sigma NP = \Sigma PN;$$

also that the product of two different T's is zero; and hence that

$$T_{a_1, a_2, \ldots, a_h} = A_{a_1, a_2, \ldots, a_h} T^2_{a_1, a_2, \ldots, a_h}.$$

2. Consider the expression PNP. If s be any substitution in one of the groups of P, $sNP = N'P$, where N' is the result of operating on N with the substitution s; hence

$$PNP = \Sigma N'P,$$

where the Σ contains $\beta_1!\,\beta_2!\,\ldots\,\beta_k!$ terms—the numbers $\beta_1, \beta_2, \ldots, \beta_k$ being the degrees of the different groups of P.

Hence, if $\lceil S \rfloor$ denotes a substitutional expression S which *operates on*, but does not multiply, the substitutions which follow it,

$$\lceil\{a_1 a_2 \ldots a_n\}\rfloor PNP = \beta_1!\,\beta_2!\,\ldots\,\beta_k!\,\lceil\{a_1 a_2 \ldots a_n\}\rfloor NP$$
$$= \beta_1!\,\beta!\,\ldots\,\beta_k!\,T_{a_1, a_2, \ldots, a_h}.$$

* In the paper referred to $T_{a_1, a_2, \ldots, a_h}$ was defined as the sum of all the *different* expressions S thus obtained. The above definition will be found more convenient.

Similarly it may be proved that

$$\lfloor\{a_1 a_2 \dots a_n\}\rceil NPN = a_1!\, a_2! \dots a_h!\, T_{a_1, a_2, \dots, a_h},$$

or, if Σ refer only to those terms which are different,

$$T_{a_1, a_2, \dots, a_h} = \Sigma PNP = \Sigma NPN.$$

3. Consider now the expression

$$P'N'P' . NP.$$

If $P'N$ is not zero, no two letters belonging to any group of P' can occur in the same group of N. Thus each of the h groups of N must contain one letter of the largest group of P'—that order h. And by similar reasoning it follows that a table may be constructed of which the rows represent the groups of N and the columns those of P'. Now

$$P'N'P' = P'N''P',$$

where N'' is obtained from N' by any substitution of P'. Such a substitution is equivalent to a change in the table giving $N'P'$ which does not alter the column in which any letter lies. Thus no alterations in the arrangement of the letters in a column of this table can affect the value of $P'N'P'$. But by such a change we can obtain the table for NP'; hence

$$P'N'P' . NP = P'NP' . NP.$$

The same argument applies to the form

$$NP'N.$$

Hence

$$P'NP' . NP = P'NP . NP.$$

But P' is transformed to P by the operation of a substitution s which is contained in N. Hence

$$P'N'P' . NP = P'NP . NP = \pm sPNP\,NP,$$

according as s is even or odd.

Now

$$T = \Sigma P'N'P',$$

and therefore

$$T . NP = (\Sigma P'N'P')\, NP,$$

where the Σ extends to those forms of P' for which $P'N \neq 0$.

But these forms of P' are those, and all those, which are obtained by operating on P with each of the substitutions of N. Hence

$$\begin{aligned} T . NP &= (\Sigma P'N'P')\, NP = (\Sigma P'NP')\, NP \\ &= (\Sigma \pm s)\, PNPNP = NPNPNP = (NP)^3. \end{aligned}$$

Further $$T = [\{a_1 a_2 \dots a_n\}] NP;$$

therefore $$T^2 = [\{a_1 a_2 \dots a_n\}] T . NP$$
$$= [\{a_1 a_2 \dots a_n\}] (NP)^2.$$

We shall proceed to show that
$$(NP)^2 = \lambda NP;$$
and hence $$T^2 = \lambda^2 T,$$
from which we obtain the coefficient
$$A = \lambda^{-2}.$$

4. If S be any substitutional expression whatever, and G be a positive symmetric group which contains all the letters of one of the groups G_1 of P, and also one letter of a group G_2 of P which is not of higher degree, then
$$NSGP = 0.$$
For $$GP = P_1 S_1,$$
where the groups of P which contain no letters of G are groups of P_1, G replaces the group G_1, and the group G_2 is replaced by that group which contains all its letters except that one contained in G. Now, when the terms $T_{a_1, a_2, \dots, a_h}$ are placed in order as explained before, it follows that P_1 belongs to an earlier T than P; and hence
$$NP_1 = 0,$$
whatever N is chosen from the terms of the particular T. Now, if s be any substitution
$$Ns = sN',$$
where N' is obtained from N by the interchange of certain of its letters, then, if
$$S = \Sigma \mu s,$$
$$NSGP = N(\Sigma \mu s) P_1 S_1 = \Sigma \mu s N' P_1 S_1 = 0.$$

5. Let the two groups of lowest order in P be
$$\{a_1 a_2 \dots a_{\beta_1}\}\{b_1 b_2 \dots b_{\beta_2}\} \quad (\beta_2 \nless \beta_1),$$
and let $$P = \{a_1 a_2 \dots a_{\beta_1}\}\{b_1 b_2 \dots b_{\beta_2}\} P_1.$$
Then $$NP . P = NP\left[1 - \left(1 + (a_1 b_1) + (a_1 b_2) + \dots + (a_1 b_{\beta_2})\right)\right] P,$$
for $$\left(1 + (a_1 b_1) + (a_1 b_2) + \dots (a_1 b_{\beta_2})\right)\{b_1 b_2 \dots b_{\beta_2}\} = \{a_1 b_1 b_2 \dots b_{\beta_2}\},$$

and, by what has just been proved,

$$NP\{a_1 b_1 b_2 \ldots b_{\beta_2}\} = 0.$$

Hence $$NP\,.\,P = NP\left[-\beta_2(a_1 b_1)\right]P$$
$$= NP\left[\beta_2\{a_1 b_1\}' - \beta_2\right]P;$$

and therefore $$NP\,.\,P = \frac{\beta_2}{\beta_2+1}NP\{a_1 b_1\}'P.$$

Let us assume then that

$$NP\,.\,P = \lambda_r NP\{a_1 b_1\}'\{a_2 b_2\}' \ldots \{a_r b_r\}'P.$$

Then

$$0 = NP\{a_1 b_1\}'\{a_2 b_2\}' \ldots \{a_r b_r\}'\big(1 + (b_1 a_{r+1}) + (b_2 a_{r+1}) + \ldots$$
$$\ldots + (b_{\beta_2} a_{r+1})\big)P$$
$$= NP\{a_1 b_1\}'\{a_2 b_2\}' \ldots \{a_r b_r\}'\big(1 + r(b_1 a_{r+1}) + (\beta_2 - r)(a_{r+1} b_{r+1})\big)P$$
$$= NP\{a_1 b_1\}'\{a_2 b_2\}' \ldots \{a_r b_r\}'\big(\beta_2 - r + 1 - (\beta_2 - r)\{a_{r+1} b_{r+1}\}'\big)P,$$

for $NP\{a_1 b_1\}'\{a_2 b_2\}' \ldots \{a_r b_r\}'(b_1 a_{r+1})P$

$$= NP(b_1 a_{r+1})\{a_{r+1} a_1\}'\{a_2 b_2\}' \ldots \{a_r b_r\}'P = 0.$$

Hence $$NP\,.\,P = \lambda_r NP\{a_1 b_1\}'\{a_2 b_2\}' \ldots \{a_r b_r\}'P$$
$$= \lambda_r \frac{\beta_2 - r}{\beta_2 - r + 1}NP\{a_1 b_1\}' \ldots \{a_{r+1} b_{r+1}\}'P;$$

and therefore

$$NP\,.\,P = \frac{\beta_2}{\beta_2+1}NP\{a_1 b_1\}'P = \ldots$$
$$= \frac{\beta_2 - \beta_1 + 1}{\beta_2 + 1}NP\{a_1 b_1\}'\{a_2 b_2\}' \ldots \{a_{\beta_1} b_{\beta_1}\}'P.$$

Let $\{c_1 c_2 \ldots c_{\beta_3}\}$ be the group of next lowest order in P. Then

$$0 = \quad NP\{a_1 b_1 c_1\}'\{a_2 b_2 c_2\}' \ldots \{a_r b_r c_r\}'\{a_{r+1} b_{r+1}\}' \ldots$$
$$\ldots \{a_{\beta_1} b_{\beta_1}\}'\{a_{r+1} c_1 c_2 \ldots c_{\beta_3}\}P$$
$$+ NP\{a_1 b_1 c_1\}'\{a_2 b_2 c_2\}' \ldots \{a_r b_r c_r\}'\{a_{r+1} b_{r+1}\}' \ldots$$
$$\ldots \{a_{\beta_1} b_{\beta_1}\}'\{b_{r+1} c_1 c_2 \ldots c_{\beta_3}\}P$$
$$= \quad NP\{a_1 b_1 c_1\}' \ldots \{a_r b_r c_r\}'\{a_{r+1} b_{r+1}\}' \ldots \{a_{\beta_1} b_{\beta_1}\}'$$
$$\times\left[1 + r(a_{r+1} c_1) + (\beta_3 - r)(a_{r+1} c_{r+1}) + 1 + r(b_{r+1} c_1)\right.$$
$$\left.+ (\beta_3 - r)(b_{r+1} c_{r+1})\right]P\,.\,\beta_3!$$
$$= NP\{a_1 b_1 c_1\}' \ldots \{a_r b_r c_r\}'\{a_{r+1} b_{r+1}\}' \ldots \{a_{\beta_1} b_{\beta_1}\}!$$
$$\times\left[2 + \beta_3 - r - (\beta_3 - r)\big(1 - (a_{r+1} c_{r+1}) - (b_{r+1} c_{r+1})\big)\right]P\,.\,\beta_3!\,.$$

But $\{a_{r+1}b_{r+1}\}'\big(1-(a_{r+1}c_{r+1})-(b_{r+1}c_{r+1})\big) = \{a_{r+1}b_{r+1}c_{r+1}\}'$.

Hence

$$NP\,\{a_1b_1c_1\}'\{a_2b_2c_2\}'\dots\{a_rb_rc_r\}'\{a_{r+1}b_{r+1}\}'\dots\{a_{\beta_1}b_{\beta_1}\}'\,P$$

$$= \frac{\beta_3-r}{\beta_3-r+2}\,NP\,\{a_1b_1c_1\}'\{a_2b_2c_2\}'\dots\{a_rb_rc_r\}'\{a_{r+1}b_{r+1}c_{r+1}\}' \times\{a_{r+2}b_{r+2}\}'\dots\{a_{\beta_1}b_{\beta_1}\}'\,P.$$

Hence

$$NP.P = \frac{\beta_2-\beta_1+1}{\beta_2+1}\,\frac{(\beta_3-\beta_1+1)(\beta_3-\beta_1+2)}{(\beta_3+1)(\beta_3+2)}\,NP\{a_1b_1c_1\}'\{a_2b_2c_2\}'\dots\dots\{a_{\beta_3}b_{\beta_3}c_{\beta_3}\}'\,P$$

$$= \frac{\beta_2-\beta_1+1}{\beta_2+1}\,\frac{(\beta_3-\beta_1+1)(\beta_3-\beta_1+2)}{(\beta_3+1)(\beta_3+2)}\,\frac{\beta_3-\beta_2+1}{\beta_3-\beta_1+1},$$

$$NP\,\{a_1b_1c_1\}'\dots\{a_{\beta_3}b_{\beta_3}c_{\beta_3}\}'\{b_{\beta_3+1}c_{\beta_3+1}\}'\dots\{b_{\beta_2}c_{\beta_2}\}'\,P.$$

Thus it is easy to see that when the letters of the r-th group (of degree β_r) are introduced into the product of negative symmetric groups a factor must be introduced outside $= \lambda_r$, where

$$\lambda_r = \frac{(\beta_r-\beta_1+r-1)!\,\beta_r!}{(\beta_r-\beta_1)!\,(\beta_2+r-1)!}\,\frac{\beta_r-\beta_2+r-2)!\,(\beta_r-\beta_1)!}{(\beta_r-\beta_2)!\,(\beta_r-\beta_1+r-2)!}\dots$$

$$\dots\frac{(\beta_r-\beta_s+r-s)!\,(\beta_r-\beta_{s-1})!}{(\beta_r-\beta_s)!\,(\beta_r-\beta_{s-1}+r-s)!}\dots\frac{(\beta_r-\beta_{r-1}+1)!\,(\beta_r-\beta_{r-2})!}{(\beta_r-\beta_{r-1})!\,(\beta_r-\beta_{r-2}+1)!}$$

$$= \frac{(\beta_r-\beta_1+r-1)(\beta_r-\beta_2+r-2)\dots(\beta_r-\beta_{r-1}+1)\,\beta_r!}{(\beta_r+r-1)!}.$$

Hence
$$NP = \frac{1}{\beta_1!\,\beta_2!\dots\beta_k!}\,NP.P$$

$$= \frac{\lambda_1}{\beta_1!}\,\frac{\lambda_2}{\beta_2!}\dots\frac{\lambda_k}{\beta_k!}\,(NP)^2$$

$$= \frac{\prod\limits_{\substack{r,s=1,\dots,k\\ r>s}}(\beta_r-\beta_s+r-s)}{\prod\limits_{r=1}^{r=k}(\beta_r+r-1)}\,(NP)^2.$$

Hence in the series

$$1 = A'_{\beta_1,\beta_2,\dots,\beta_k}\,T'_{\beta_1,\beta_2,\dots,\beta_k},$$

where $\beta_1, \beta_2, \dots, \beta_k$ are the degrees of the positive symmetric groups, such that

$$\beta_1 \not< \beta_2 \not< \beta_3 \not< \dots \not< \beta_k,$$

the coefficient $$A'_{\beta_1, \beta_2, \ldots, \beta_k} = \left(\frac{\prod\limits_{r,s} (\beta_r - \beta_s + r - s)}{\prod\limits_{r} (\beta_r + r - 1)!} \right)^2.$$

6. The numbers $\beta_1, \beta_2, \ldots, \beta_k$ are here in ascending order of magnitude: we have, however, been taking the suffixes in descending order; then, if $$\gamma_1 = \beta_k, \; \gamma_2 = \beta_{k-1}, \; \ldots, \; \gamma_k = \beta_1,$$

$$A''_{\gamma_1, \gamma_2, \ldots, \gamma_k} = A'_{\beta_1, \beta_2, \ldots, \beta_k} = \left(\frac{\prod\limits_{r,s} (\gamma_r - \gamma_s - r + s)}{\prod\limits_{r} (\gamma_r + k - r)!} \right)^2.$$

In this series $$1 = \Sigma A''_{\gamma_1, \gamma_2, \ldots, \gamma_k} T''_{\gamma_1, \gamma_2, \ldots, \gamma_k}.$$

Change the sign of every transposition; then

$$T''_{\gamma_1, \gamma_2, \ldots, \gamma_k}$$

becomes $$T_{\gamma_1, \gamma_2, \ldots, \gamma_k},$$

and, since the constants are unaffected by this change of sign, we obtain $$A_{a_1, a_2, \ldots, a_h} = \left(\frac{\prod\limits_{r,s} (a_r - a_s - r + s)}{\prod\limits_{r} (a_r + h - r)!} \right)^2$$

for the value of the constants in the series

$$1 = \Sigma A_{a_1, a_2, \ldots, a_h} T_{a_1, a_2, \ldots, a_h}.$$

It is to be observed that $T_{a_1, \ldots, a_h}$ might be equally well defined by its positive symmetric groups: the value of the constant must be the same with this definition. Now the degrees of the positive symmetric groups of $T_{a_1, a_2, \ldots, a_h}$ are as follows: there are a_h groups of degree h; $a_{h-1} - a_h$ of degree $h-1$; $a_{h-2} - a_{h-1}$ of degree $h-2$, and so on; finally, there are $a_1 - a_2$ groups of degree unity. If these numbers be substituted for $a_1, a_2, \ldots, a_h$, it can be verified that the resulting coefficient $= A_{a_1, a_2, \ldots, a_h}$.

7. The following particular results of this investigation should be noticed.

If NP be a term of T, the coefficient in the series

$$1 = \Sigma A_{a_1, \ldots, a_h} T_{a_1, \ldots, a_h}$$

of T being A; then
$$(NP)^2 = A^{-1}NP,$$
$$T.NP = A^{-1}NP,$$
$$T.PN = A^{-1}PN.$$

Again,
$$TN = (\Sigma P'N'P')N$$
$$= \Sigma P'NP'N$$
$$= NPNPN$$
$$= A^{-1}NPN.$$

Similarly,
$$T.P = A^{-1}PNP.$$

8. If P be expressed as a sum of substitutions, the coefficient of each is unity. Similarly, if N be expressed as a sum of substitutions, the coefficient of each is ± 1. Moreover, if two letters appear in the same cycle of one of the substitutions of P, they cannot appear in the same cycle of a substitution of N. Hence, if s be a substitution of P, s^{-1} cannot appear in N unless $s = 1$. The coefficient of the identical substitution in NP must then be 1.

Again, the coefficient of every substitution in NP is ± 1. For, let t be any substitution of this product, and let it arrive by taking the substitution s_1 of N with σ_1 of P, so that
$$s_1\sigma_1 = t.$$

Suppose that it occurs a second time in the sum, as, say, $s_2\sigma_2$. Then
$$s_1\sigma_1 = s_2\sigma_2,$$
and therefore
$$s_2^{-1}s_1\sigma_1\sigma_2^{-1} = 1.$$
But $s_2^{-1}s_1$ is a substitution of N, and $\sigma_1\sigma_2^{-1}$ is a substitution of P; hence
$$s_2^{-1}s_1 = 1 = \sigma_1\sigma_2^{-1},$$
and therefore
$$s_1 = s_2, \quad \sigma_1 = \sigma_2.$$
The substitution t then can only occur once in NP. The same remark applies to PN.

Now
$$T = \ulcorner\{(a_1 a_2 \ldots a_n)\}\urcorner NP:$$
the coefficient of the identical substitution in T must then be $n!$. Hence we obtain the identity
$$\frac{1}{n!} = \Sigma A_{a_1, a_2, \ldots, a_n}$$
by equating the coefficients of the identical substitution. Remember-

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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

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ing that the operations are made in the order expressed by the equation

$$s_1 s_2 F = s_1 [s_2 F],$$

we observe that, if S be any substitutional expression,

$$\sigma S \sigma^{-1} = \lceil\!\lfloor \sigma \rfloor\!\rceil S.$$

Hence, if

$$S = \lceil\!\lfloor \{a_1 a_2 \dots a_n\} \rfloor\!\rceil S',$$

$$\sigma S \sigma^{-1} = S$$

or

$$\sigma S = S \sigma.$$

It follows from this that any one of the expressions T is commutative with any single substitution; and therefore with any substitutional expression whatever which contains no letters which do not appear in T.

II. *Relations between different Forms NP.*

9. If s be any substitution, we obtain by means of the T series

$$s = \Sigma A_{a_1, a_2, \dots, a_h} T_{a_1, a_2, \dots, a_h} s\,;$$

thus every substitution can be expressed in terms of forms NPs (or PNs).

Between forms NP certain linear relations occur. To discuss them it is necessary to employ a notation which will completely define the form under discussion.

We will write

$$\left\{\begin{matrix} a_{1,1}\, a_{1,2} \cdots\cdots\cdots\cdots\cdots a_{1,\beta_1} \\ a_{2,1}\, a_{2,2} \cdots\cdots\cdots a_{2,\beta_2} \\ \dots \quad \dots \quad \dots \quad \dots \\ a_{k,1}\, a_{k,2} \dots a_{k,\beta_k} \end{matrix}\right\} \equiv \{a_{1,1}\, a_{1,2} \dots a_{1,\beta_1}\}\{a_{2,1} \dots a_{2,\beta_2}\} \dots$$

$$\dots\{a_{k,1}\, a_{k,2} \dots a_{k,\beta_k}\}\{a_{1,1}\, a_{2,1} \dots a_{k,1}\}' \dots \dots \{a_{1,\beta_1}\}'$$

$$= PN.$$

Here the rows are taken to define the positive symmetric groups, while the columns define the negative symmetric groups.

Again, we will write

$$\left\{\begin{matrix} a_{1,1}\, a_{1,2} \cdots\cdots\cdots a_{1,\alpha_1} \\ a_{2,1}\, a_{2,2} \dots a_{2,\alpha_2} \\ \dots \quad \dots \quad \dots \\ a_{h,1} \dots a_{h,\alpha_h} \end{matrix}\right\}' \equiv \{a_{1,1}\, a_{1,2} \dots a_{1,\alpha_1}\}'\, \{a_{2,1} \dots a_{2,\alpha_2}\}' \dots$$

$$\dots \{a_{h,1} \dots a_{h,\alpha_h}\}'\, \{a_{1,1}\, a_{2,1} \dots a_{h,1}\} \dots \{a_{1,\alpha_1}\}$$

$$= NP,$$

where the rows define the negative symmetric groups, the columns the positive symmetric groups. The notation used for a single positive or negative symmetric group is, it will be noticed, a particular case of that now introduced.

10. The product $(s > r)$

$$\{a_{r,1} \dots a_{r,\beta_r} a_{s,\lambda}\} \left\{ \begin{matrix} a_{1,1}\, a_{1,2} \dots\dots\dots a_{1,\beta_1} \\ a_{2,1} \dots\dots\dots\dots \\ \dots \quad \dots \quad \dots \\ a_{r,1} \dots\dots\dots a_{r,\beta_r} \\ \dots \quad \dots \quad \dots \\ a_{h,1} \dots a_{h,\beta_h} \end{matrix} \right\}$$

is zero. This follows at once from § 4.

Similarly, the product $(s > r)$

$$\left\{ \begin{matrix} a_{1,1}\, a_{1,2} \dots a_{1,r} \dots a_{1,\beta_1} \\ a_{2,1} \quad \dots\dots \\ \dots\dots\dots\dots \; a_{\alpha_n, r} \\ a_{h,1} \quad \dots\dots \quad a_{h,\beta_h} \end{matrix} \right\} \{a_{1,r}\, a_{2,r} \dots a_{\alpha_n, r}\, a_{\lambda, s}\}' = 0.$$

All relations between forms PNS may be linearly obtained from relations of these two kinds and the obvious relations

$$\sigma P N \sigma^{-1} = \lfloor \sigma \rceil PN,$$

$$(a_{r,\lambda}\, a_{r,\mu})\, PN = PN,$$

$$PN\, (a_{\lambda, r}\, a_{\mu, r}) = -PN.$$

It is to be observed that, since

$$\{a_{r,1} \dots a_{r,\beta_r} a_{s,\lambda}\}$$
$$= \left[1 + (a_{s,\lambda}\, a_{r,1}) + (a_{s,\lambda}\, a_{r,2}) + \dots + (a_{s,\lambda}\, a_{r,\beta_r})\right] \{a_{r,1} \dots a_{r,\beta_r}\}$$

and

$$\{a_{1,r}\, a_{2,r} \dots a_{\alpha_r, r}\, a_{\lambda, s}\}'$$
$$= \{a_{1,r}\, a_{2,r} \dots a_{\alpha_r, r}\}\left[1 - (a_{1,r}\, a_{\lambda,s}) - \dots - (a_{\lambda_r, r}\, a_{\lambda, s})\right],$$

[redacted]

the above identities may be written

$$\left[1+(a_{s,\lambda}\,a_{r,1})+(a_{s,\lambda}\,a_{r,2})+\ldots+(a_{s,\lambda}\,a_{r,\beta_r})\right]PN=0$$

and $$PN\left[1-(a_{1,r}\,a_{\lambda,s})-(a_{2,r}\,a_{\lambda,s})-\ldots-(a_{\alpha_r,r}\,a_{\lambda,s})\right]=0.$$

11. Before establishing the theorem just ennunciated it is necessary to prove that, if s be any substitution,

$$PsPN=\lambda\,PN$$

and $$PNsN=\mu\,PN,$$

where λ and μ are numerical constants. Let us suppose s written as a product of cycles; then if any cycle σ contains two letters b_1, b_2 out of the same group of P,

$$P\sigma=P\sigma'\sigma'',$$

where $\sigma'\sigma''$ are two independent cycles all the letters of which appear in σ, but which are such that b_1 occurs in σ' and b_2 in σ''.

To prove this we observe that

$$P\sigma=P\,(b_1b_2)\,\sigma,$$

and consider the following cases.

(i.) Let $$\sigma=(ab_1b_2c\ldots);$$

then $$(b_1b_2)\,\sigma=(ab_2c\ldots)=\sigma'',\quad \sigma'=1.$$

(ii.) Let $$\sigma=(ab_1c\ldots db_2ef\ldots);$$

then $$(b_1b_2)\,\sigma=(ab_2ef\ldots)(b_1c\ldots d).$$

Let the greatest group of P in which any of the letters of σ appear be $\{a_1a_2\ldots a_r\}$. Suppose that of these letters $a_1, a_2, \ldots, a_h$ appear in s, necessarily all in different cycles. The cycle in which a_h appears we will call σ_1, and let

$$s=\sigma_1\sigma'.$$

Now σ_1 may be written in the form

$$\sigma_1'\,(a_hb),$$

where σ_1' is a cycle which contains the letter b, but not the letter a_h, and where b belongs to a group of P whose degree is $\not> r$.

Then

$$PsPN = P\sigma'\sigma_1'(a_h b)\,PN$$

$$= \frac{1}{r+1-h}P\sigma'\sigma_1'\left[(a_h b)+(a_{h+1}b)+\ldots+(a_r b)\right]PN$$

$$= \frac{1}{r+1-h}P\sigma'\sigma_1'\left[-\big(1+(a_1 b)+(a_2 b)+\ldots+(a_{h-1}b)\big)\right.$$

$$\left.+\big(1+(a_1 b)+(a_2 b)+\ldots+(a\,b)\big)\right]PN$$

$$= \frac{-1}{r+1-h}P\sigma'\sigma_1'\left[1+(a_1 b)+(a_2 b)+\ldots+(a_{h-1}b)\right]PN,$$

owing to the relation satisfied by PN,

$$= \frac{-1}{r+1-h}P\,[\Sigma s']\,PN,$$

where s' affects fewer letters than s.

Proceeding thus step by step, we obtain

$$Ps\,PN = \lambda\,PN,$$

and similarly $$PNsN = \mu\,PN,$$

where λ and μ are numerical.

12. We proceed now to show that the relations of § 10 embrace all relations satisfied by PN. In the first place, it is to be noticed that the only relations used as yet are those of § 10; whether in proving that

$$TT' = 0,\quad (PN)^2 = A^{-1}PN,\quad TPN = A^{-1}PN,$$

or in the theorem of the last paragraph.

Let $$\Sigma\lambda_i P_i N_i s_i = 0 \qquad \text{(I.)}$$

be any such relation.

If all the forms $P_i N_i$ do not belong to the same member of the series

$$1 = \Sigma AT,$$

let T be a member which contains some of them.

Then multiply the equation (I.) by T on the left-hand side. If $P_i N_i$ belongs to T, then

$$TP_i N_i s_i = A^{-1}P_i N_i s_i.$$

Otherwise $$TP_i N_i s = 0.$$

Hence we obtain $A^{-1}\{\Sigma\lambda_j P_j N_j s_j\} = 0,$

where the terms of this sum are just those terms of (I.) which belong to T. Hence no relation can connect forms belonging to different T's.

Consider the relation $\Sigma\lambda_j P_j N_j s_j = 0,$ (II.)

where each term belongs to T. Let PN be any definite term of T; then, by means of certain interchanges of the letters, we may change PN into $P\ N_j$; in other words,

$$P_j N_j = \sigma_j PN\sigma_j^{-1}.$$

The relation may then be written

$$\Sigma\lambda_j\sigma_j PN\sigma_j^{-1}s_j = 0.$$

Multiply this on the right-hand side by N; then

$$\Sigma\mu_j\,\sigma_j PN = 0, \tag{III.}$$

where $$\lambda_j PN\sigma_j^{-1}N = \mu_j PN.$$

Multiply (III.) on the left-hand side by P, and we obtain

$$\Sigma\nu_j PN = 0;$$

and hence $$\Sigma\nu_j = 0,$$

where $$\mu_j P\sigma\ PN = \nu_j PN.$$

Let us return to the original equation; then

$$\Sigma\lambda_j P_j N_j s_j AT = \Sigma\lambda_j P_j N_j s_j = 0;$$

but $$T = \Sigma NPN.$$

The left-hand side of this relation is then equal to a sum of expressions $$\Sigma\lambda_j\sigma_j\ PN\sigma_j^{-1}s_j NPN,$$

each of which is of the form $\Sigma\mu_j\sigma_j\,PN$,

since $$(PN)^2 = A^{-1}PN.$$

But each of these expressions is separately zero by (III.). Hence the original equation is expressible in terms of relations (III.), by means of the relations of § 10.

Consider then the relation

$$AT\,\Sigma\mu_j\sigma_j\,PN = \Sigma\mu_j\sigma\ PN = 0.$$

Take any term $P'N'P'$ of T; then

$$P'N'P' = \sigma' PNP\sigma'^{-1}.$$

Hence this relation is equal to a sum of relations of the form

$$\sigma' PNP\sigma'^{-1}\,[\Sigma \mu_j \sigma_j]\,PN = 0,$$

and this reduces, by § 11, to

$$[\Sigma \nu_j]\,\sigma' PN = 0.$$

The original relation may then be reduced to a merely numerical identity, by the use of relations derived from those of § 10. Hence no relation can exist which is independent of these.

13. Consider the expression of any substitution σ in terms of the forms PNs. It is obtained by means of the identity

$$\sigma = \Sigma A T\sigma.$$

To simplify this expression it is necessary to consider the product of PN by any substitution.

Let $$PN = \left\{\begin{matrix} a_{1,1}\, a_{1,2} \;\cdots\cdots\cdots\; a_{1,\beta_1} \\ a_{2,1}\, a_{2,2} \;\cdots\; a_{2,\beta_2} \\ \cdots \quad \cdots \quad \vdots \\ a_{k,1} \cdots a_{k,\beta_k} \end{matrix}\right\}.$$

(i.) $\sigma = (a_{\lambda,s}\, a_{\mu,s})$; then

$$PN\sigma = -PN.$$

(ii.) $\sigma = (a_{r,\lambda}\, a_{r,\mu})$; then

$$PN\sigma = \llcorner \sigma \urcorner\, PN.$$

(iii.) $\sigma = (a_{r,\lambda}\, a_{s,\mu})$; then

$$PN\sigma = P\sigma N',$$

where $$N' = \llcorner (a_{r,\lambda}\, a_{s,\mu}) \urcorner\, N.$$

Now $a_{r,\lambda}$, $a_{r,\mu}$ appear in the same group of N'; they also appear in the same group of P.

Hence $$PN\,(a_{r,\lambda}\, a_{s,\mu}) = P\,(a_{r,\lambda}\, a_{s,\mu})\, N' = -P\,(a_{r,\mu}\, a_{s,\mu})\, N'$$
$$= -PN''\,(a_{r,\mu}\, a_{s,\mu}),$$

where $$N'' = \llcorner (a_{r,\mu}\, a_{s,\mu}) \urcorner\, N'$$
$$= \llcorner (a_{r,\mu}\, a_{s,\mu})(a_{r,\lambda}\, a_{s,\mu}) \urcorner\, N$$
$$= \llcorner (a_{r,\mu}\, a_{r,\lambda}) \urcorner\, N,$$

since $$\lfloor (a_{r,\mu}\, a_{s,\mu}) \rceil N = N.$$

Hence $$\lfloor \{a_{r,\lambda}\, a_{r,\mu}\} \rceil PN\, (a_{r,\lambda}\, a_{s,\mu}) = 0.$$

Similarly, $$\lfloor \{a_{s,\lambda}\, a_{s,\mu}\} \rceil PN\, (a_{r,\lambda}\, a_{s,\mu}) = 0.$$

These identities presuppose that

$$\mu \not> \beta_r \quad \text{and} \quad \lambda \not> \beta_s;$$

since we know that $\mu \not> \beta_s$ and $\lambda \not> \beta_r$, one of these identities exists in every case.

14. It was proved in § 11 that, if s be any substitution,

$$Ns = \pm Ns',$$

where s' is a product of cycles no one of which contains a pair of letters from the same group of N. Hence, when considering a term $PN\sigma$, we may suppose σ to be a product of such cycles. In the same way we may break up the cycles of σ and obtain

$$PN\sigma = PN'\sigma',$$

where the cycles of σ' are such that no two letters either in the same row or the same column of the expression PN' occur in the same cycle.

The two equations just written down enable us to still further reduce σ.

15. If σ is not a substitution of PN, then

$$(\lfloor \sigma \rceil N)\, P = 0.$$

For $$N'P = 0,$$

if a group of N' contains two letters of one of the groups of P. But, if

$$N'P \neq 0,$$

we have seen that there is some substitution σ' of P such that

$$\lfloor \sigma' \rceil N = N'.$$

Hence, if $$\lfloor \sigma \rceil N = N',$$

$$\sigma = \sigma'\sigma'',$$

where σ'' is a substitution which leaves N unaltered, and hence belongs to N (unless N contains two or more equal groups, and these are interchanged *in toto;* this may be supposed done by a

substitution of P and included in σ'). Therefore, if

$$(\lfloor\sigma\rceil N)\,P \neq 0,$$

σ must be a substitution of PN.

16. Consider $PN\sigma^{-1}$, where σ is not a substitution of PN,

$$PN\sigma^{-1} = P\sigma^{-1}\lfloor\sigma\rceil N.$$

Hence $$(PN\sigma^{-1})^2 = P\sigma^{-1}(\lfloor\sigma\rceil N)\,P\sigma^{-1}\lfloor\sigma\rceil N = 0.$$

Hence every such expression $PN\sigma^{-1}$ is a solution of the substitutional equation

$$S^2 = 0,$$

where S is an unknown substitutional expression.

If, however, σ is a substitution of PN, then

$$\sigma^{-1} = \sigma''^{-1}\sigma'^{-1},$$

where σ''^{-1} is a substitution of N; and σ'^{-1} is a substitution of P, and therefore $$PN\sigma^{-1} = \pm PN'.$$

Any substitutional expression can therefore be linearly expressed in terms of forms PN, whose squares are the forms themselves multiplied by a constant; and of forms $PN\sigma$ whose squares are zero.

Consider the product of two different forms PN belonging to the same T. Let these be PN and $P'N'$.

If $N'P = 0$, then

$$P'N'PN = 0.$$

Suppose that $N'P$ is not zero. If $P'N \neq 0$, there is a substitution σ of N such that

$$\lfloor\sigma\rceil P' = P.$$

Hence $$P'N'PN = \pm P'N'\sigma P'N.$$

But, unless σ^{-1} is a substitution of $P'N'$, we must have

$$P'N'\sigma P' = 0.$$

Hence $\sigma = \sigma_1\sigma_2$, where σ_1 is a substitution of N', and σ_2 one of P'.

Therefore
$$\begin{aligned} P'N'PN &= \pm P'N'\sigma P'N \\ &= \pm P'N'\sigma_1\sigma_2 P'N \\ &= \pm P'N'P'N = \pm P'NP'N \\ &= \pm A^{-1}P'N. \end{aligned}$$

If $P'N' = 0$, there exists a substitution σ such that

$$\sigma^{-1} PN = P'N'.$$

Hence $$P'N'PN = \sigma PN\sigma^{-1}PN.$$

And here $PN\sigma^{-1}P = 0$, unless $\sigma = \sigma_1\sigma_2$, where σ_1 is a substitution of P, and σ_2 is a substitution of N; so that

$$\sigma^{-1} = \sigma_2^{-1}\sigma_1^{-1}.$$

Therefore
$$\begin{aligned} P'N'PN &= \sigma PN\sigma_2^{-1}\sigma_1^{-1}PN \\ &= \pm\,\sigma PNPN = \pm\,A^{-1}\sigma PN = \pm\,A^{-1}P'\sigma N. \end{aligned}$$

Similarly, $$(P'N's')(PNs) = 0$$

or
$$\begin{aligned} &= s'P''N''PNs \\ &= \pm\,A^{-1}s'P''\sigma Ns \\ &= \pm\,A^{-1}P's'\sigma Ns. \end{aligned}$$

17. To illustrate the last few paragraphs, we will consider one or two special forms PN.

Let $$PN = \left\{\begin{matrix} a_1 a_2 \\ a_3 \end{matrix}\right\};$$

then
$$\begin{aligned} \left\{\begin{matrix} a_1 a_2 \\ a_3 \end{matrix}\right\}(a_2a_3) &= \{a_1a_2\}\{a_1a_3\}'(a_2a_3) \\ &= \{a_1a_3\}(a_2a_3)\{a_1a_2\}' \\ &= -\{a_1a_2\}(a_1a_3)\{a_1a_2\}' \\ &= -\left\{\begin{matrix} a_2 a_1 \\ a_3 \end{matrix}\right\}(a_1a_3). \end{aligned}$$

Again, $$\left\{\begin{matrix} a_1 a_2 \\ a_3 \end{matrix}\right\}\left[1-(a_2a_1)-(a_2a_3)\right] = 0;$$

hence
$$\begin{aligned} \left\{\begin{matrix} a_1 a_2 \\ a_3 \end{matrix}\right\}(a_2a_3) &= \left\{\begin{matrix} a_1 a_2 \\ a_3 \end{matrix}\right\}\left[1-(a_1a_2)\right] \\ &= \left\{\begin{matrix} a_1 a_2 \\ a_3 \end{matrix}\right\} - \left\{\begin{matrix} a_2 a_1 \\ a_3 \end{matrix}\right\}. \end{aligned}$$

Again
$$\begin{aligned} 0 &= \left[1+(a_3a_1)+(a_3a_2)\right]\left\{\begin{matrix} a_1 a_2 \\ a_3 \end{matrix}\right\} \\ &= \left\{\begin{matrix} a_1 a_2 \\ a_3 \end{matrix}\right\} - \left\{\begin{matrix} a_3 a_2 \\ a_1 \end{matrix}\right\} + \left\{\begin{matrix} a_1 a_3 \\ a_2 \end{matrix}\right\}(a_2a_3) \\ &= \left\{\begin{matrix} a_1 a_2 \\ a_3 \end{matrix}\right\} - \left\{\begin{matrix} a_3 a_2 \\ a_1 \end{matrix}\right\} + \left\{\begin{matrix} a_1 a_3 \\ a_2 \end{matrix}\right\} - \left\{\begin{matrix} a_3 a_1 \\ a_2 \end{matrix}\right\}, \end{aligned}$$

hence $\left\{\begin{array}{l} a_1 a_2 \\ a_3 \end{array}\right\} + \left\{\begin{array}{l} a_1 a_3 \\ a_2 \end{array}\right\} = \left\{\begin{array}{l} a_3 a_1 \\ a_2 \end{array}\right\} + \left\{\begin{array}{l} a_3 a_2 \\ a_1 \end{array}\right\} = \left\{\begin{array}{l} a_2 a_1 \\ a_3 \end{array}\right\} + \left\{\begin{array}{l} a_2 a_3 \\ a_1 \end{array}\right\}$

$$= \tfrac{1}{3} T_{2,1}.$$

Let $$PN = \left\{\begin{array}{l} a_1 a_3 a_4 \ldots a_{n-1} \\ a_2 a_n \end{array}\right\};$$

then $$PN\left[1-(a_3 a_1)-(a_3 a_2)\right] = 0;$$

hence $\left\{\begin{array}{l} a_1 a_3 a_4 \ldots a_{n-1} \\ a_2 a_n \end{array}\right\} (a_3 a_2)$

$$= \left\{\begin{array}{l} a_1 a_3 a_4 \ldots a_{n-1} \\ a_2 a_n \end{array}\right\} - \left\{\begin{array}{l} a_3 a_1 a_4 \ldots a_{n-1} \\ a_2 a_n \end{array}\right\}.$$

Similarly $\left\{\begin{array}{l} a_1 a_3 a_4 a_5 \ldots a_{n-1} \\ a_2 a_n \end{array}\right\} (a_4 a_2)$

$$= \left\{\begin{array}{l} a_1 a_3 a_4 a_5 \ldots a_{n-1} \\ a_2 a_n \end{array}\right\} - \left\{\begin{array}{l} a_4 a_3 a_1 a_5 \ldots a_{n-1} \\ a_2 a_n \end{array}\right\}.$$

Again $$\left[1+(a_2 a_1)+(a_2 a_3)+\ldots+(a_2 a_{n-1})\right] PN = 0;$$

hence

$$\left\{\begin{array}{l} a_1 a_3 a_4 a_5 \ldots a_{n-1} \\ a_2 a_n \end{array}\right\} + \left\{\begin{array}{l} a_1 a_2 a_4 a_5 \ldots a_{n-1} \\ a_3 a_n \end{array}\right\} + \left\{\begin{array}{l} a_1 a_3 a_2 a_5 \ldots a_{n-1} \\ a_4 a_n \end{array}\right\} + \ldots$$

$$= \left\{\begin{array}{l} a_2 a_3 a_4 a_5 \ldots a_{n-1} \\ a_1 a_n \end{array}\right\} + \left\{\begin{array}{l} a_2 a_1 a_4 a_5 \ldots a_{n-1} \\ a_3 a_n \end{array}\right\} + \left\{\begin{array}{l} a_2 a_3 a_1 a_5 \ldots a_{n-1} \\ a_4 a_n \end{array}\right\} + \ldots;$$

that is

$$\left[\{a_1 a_2\}' \left[1+(a_2 a_3)+(a_2 a_4)+\ldots+(a_2 a_{n-1})\right]\right] \left\{\begin{array}{l} a_1 a_3 a_4 \ldots a_{n-1} \\ a_2 a_n \end{array}\right\} = 0.$$

In general it is to be observed that, if a, b be two letters such that the groups of N in which they appear are each of degree <3, or else such that the groups of P in which they appear are each of degree <3, then

$$PN(ab) = \pm P'N' - P''N''.$$

III. *The Product of a Symmetric Group into PN.*

18. First consider the product

$$\{a_1 a_2\} \left\{\begin{array}{l} a_1 b_1 c_1 \ldots \\ a_2 b_2 \ldots\ldots \\ a_3 \ldots \\ \ldots \end{array}\right\}.$$

Since $$[1+(a_2a_1)+(a_2b_1)+(a_2c_1)+\dots]\left\{\begin{matrix} a_1b_1c_1\dots \\ a_2b_2c_2\dots \\ \dots\ \dots\ \dots \end{matrix}\right\}=0,$$

therefore

$$\{a_1a_2\}\left\{\begin{matrix} a_1b_1\dots \\ a_2b_2\dots \\ \dots\ \ \dots \end{matrix}\right\}=-\left\{\begin{matrix} a_1a_2c_1\ \dots \\ b_1b_2c_2\ \dots \\ a_3b_3\dots\dots \\ \dots\ \dots\ \dots \end{matrix}\right\}(a_2b_1)-\left\{\begin{matrix} a_1b_1a_2\dots \\ c_1b_2c_2\ \dots \\ a_3b_3\dots\dots \\ \dots\ \dots\ \dots \end{matrix}\right\}(a_2c_1),$$

every term of which is of the form $PN\sigma$, where $\{a_1a_2\}$ is a factor of one of the groups of P.

Similarly $$\{a_1b_2\}\left\{\begin{matrix} a_1b_1c_1\dots \\ a_2b_2c_2\dots \\ \dots\ \dots\ \dots \end{matrix}\right\}=-\Sigma PN\sigma,$$

where in each term $\{a_1b_2\}$ is a factor of P.

More generally the product

$$\{a_{1,1}a_{1,2}\dots a_{1,r}a_{2,s}\}\left\{\begin{matrix} a_{1,1}\,a_{1,2}\dots a_{1,\beta_1} \\ a_{2,1}\,a_{2,2}\dots \\ a_{3,1}\dots \\ \dots\ \dots \end{matrix}\right\}$$

$$=r!\,[1+(a_{2,s}a_{1,1})+(a_{2,s}a_{1,2})+\dots+(a_{2,s}a_{1,r})]\left\{\begin{matrix} a_{1,1}\,a_{1,2}\dots a_{1,\beta_1} \\ a_{2,1}\,a_{2,2}\dots \\ a_{3,1}\dots \\ \dots\ \dots \end{matrix}\right\}$$

$$=-r!\,[(a_{2,s}a_{1,r+1})+\dots+(a_{2,s}a_{1,\beta_1})]\left\{\begin{matrix} a_{1,1}\,a_{1,2}\dots a_{1,\beta_1} \\ a_{2,1}\,a_{2,2}\dots \\ a_{3,1}\dots \\ \dots\ \dots \end{matrix}\right\}$$

$$=-r!\,\Sigma P'N'\sigma,$$

in every term of which $\{a_{1,1}a_{1,2}\dots a_{1,r}a_{r,s}\}$ is a factor of P'. And, further, P' is only changed in that one of the letters $a_{1,t}\,(t>r)$, in the group $\{a_{1,1}a_{1,2}\dots a_{1,\beta_1}\}$ which contains $\{a_{1,1}\dots a_{1,r}\}$ as a factor, is interchanged with $a_{2,s}$; and this is done in all possible ways.

We have supposed that the letters $a_{1,1}, a_{1,2}, \dots, a_{1,r}$ lie in the greatest group of P; this is unnecessary, provided $a_{2,s}$ does not lie in a group of degree greater than that of the group which contains the letters $a_{1,1}, a_{1,2}, \dots$.

If, however, $\beta_1 \ngtr r$, then the above product is zero.

19. In the same way it may be shown that the product of a positive symmetric group $\{a_1 a_2 \dots a_r\}$ into any form PN may be expressed in the form $\Sigma\lambda P'N'\sigma$, where $\{a_1 a_2 \dots a_r\}$ is a factor of every P', and, moreover, is a factor of that group of P' which is of degree β, β being the degree of the greatest group of P which contains one of the letters $a_1, a_2, \dots, a_r$.

For, if a_1 is the letter which appears in the group of P degree β, then

$$\{a_1 a_2 \dots a_n\} PN$$
$$= [1+(a_n a_1)+(a_n a_2)+\dots+(a_n a_{n-1})][1+(a_{n-1}a_1)+\dots+(a_{n-1}a_{n-2})]\dots$$
$$\dots[1+(a_2 a_1)]\,PN.$$

We may proceed exactly as in the last paragraph at each step, and the result follows. We observe further that the expressions P' are just those expressions which may be obtained by filling up the places vacated by $a_2, \dots, a_r$ when these letters are moved into the group degree β in all possible ways by the remaining letters of this group.

IV. *Application to the Theory of Invariants.*

As already stated, the last three sections of this paper deal with an application of substitutional analysis to modern algebra. First, any homogeneous rational integral function F of the coefficients of certain binary quantics is considered. By the introduction of the ordinary symbolical notation this may be represented as a function $H_1\phi$; where ϕ is a function of certain sets of variables, there being q variables in each set, and H_1 is a substitutional expression denoting the fact that certain of the sets of variables refer to the same quantic. By polarization and the introduction of new sets of variables we may obtain from $H_1\phi$ a function $f = HGf_1$, which is linear in each of n sets of variables, there being q variables in each set, where G is a substitutional expression denoting the fact that certain of these sets are equivalent—*i.e.*, combine to replace a single set of $H_1\phi$—and H is the substitutional equivalent of H_1. This process was explained in § 12 of my former paper, and it was there shown that f is a function equivalent to $H_1\phi$.

Here it is proved that $HGPNf_1$ is a linear function of the coefficients of a certain concomitant of the quantics under consideration, and hence that the series $1 = \Sigma AT$ enables us to express f, and therefore F, as a linear function of coefficients of concomitants of the quantics; for
$$f = HGf_1 = HG\,(\Sigma AT)\,f_1.$$

Hence any rational integral function of the coefficients of certain q-ary quantics may be expressed linearly in terms of coefficients of concomitants of these quantics.

By the same process, taking F to be a concomitant, a proof of the fundamental theorem of symbolical algebra is obtained; viz., that F may be expressed as a sum of symbolical products, the factors being of certain definite forms.

Incidentally it appears that the function of the coefficients of transformation which appears as a factor of a concomitant after transformation must be a power of the modulus of transformation.

Further, if $HGPNf_1$ is a concomitant, then it is completely defined —except for a numerical factor—by the substitutional expression $HGPN$. Thus a certain set of concomitants is obtained in terms of which every concomitant can be linearly expressed. This set has the advantage that the linear relations between members of the set may be all simply obtained; they are given by the relations of § 10.

In Section V. the invariants of a single binary quantic are discussed from this point of view. It is shown that each of the members of the fundamental set is completely defined by certain $n+1$ numbers, where n is the order of the quantic considered. Calling these numbers $a_0, a_1, a_2, \ldots, a_n$, the invariant is written

$$f(a_0, a_1, \ldots, a_n).$$

If δ is the degree, and w the weight, of this invariant, then the relations

$$a_0+a_1+\ldots+a_n = \delta,$$

$$a_1+2a_2+\ldots+na_n = a_{n-1}+2a_{n-2}+\ldots+na_0 = w$$

must be satisfied.

The greater part of the section deals with the linear relations between these invariants; as an example it is proved that there can be no *gauche* invariant of the quintic of degree less than 18.

The discussion in this paper is limited entirely to the question of linear independence: to make the method applicable to the question of reducibility it would be necessary to investigate the product of two concomitants of the forms considered.

There is one case, however, in which such an investigation is not required—that of a single quadratic in any number of variables. This is the subject of Section VI.: it is there shown that for any given degree and orders in the different kinds of variables there is only one linearly independent concomitant; hence, this is reducible

or otherwise according as there is or is not a product of concomitants having the same degree and orders.

20. Let F be any rational integral homogeneous function of the coefficients of certain q-ary quantics, and $f = HGf_1$ the function equivalent to it, which is obtained in the manner explained above, and is linear in each of n sets of q-ary variables. These sets of variables will be denoted by the letters $a_1, a_2, \ldots, a_n$. When it is necessary to speak of the variables of the set a_r, they will be written $a_{r,1}, a_{r,2}, \ldots, a_{r,q}$.

The substitutions employed will be those of the symmetric group of the letters $a_1, a_2, \ldots, a_n$; *i.e.*, they will interchange entire sets.

The expression G denotes the fact that certain of these sets correspond to the same quantic: thus, if the set $a_1, a_2, \ldots, a_r$ correspond to a single quantic of order r, then $\{a_1 a_2 \ldots a_r\}$ is one of the factors of G. The expression H was defined as the substitutional equivalent to the expression H_1, which occurs in the result $H_1\phi$ of introducing the symbolical notation in F. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ represent the sets of variables in $H_1\phi$; then, if F is of degree s in the coefficients of a certain quantic order r, s of these sets belong to this quantic. Suppose that these are the sets $\alpha_1, \alpha_2, \ldots, \alpha_s$; then H_1 has the factor $\{\alpha_1 \alpha_2 \ldots \alpha_s\}$. To each of these sets in $H_1\phi$, r sets of f correspond; for the sake of argument the sets corresponding to α_h will be written

$$a_{(h-1)r+1},\ a_{(h-1)r+2},\ \ldots,\ a_{hr}.$$

Then the factor of H corresponding to the factor $\{\alpha_1 \alpha_2 \ldots \alpha_s\}$ of H_1 may be written $\{\alpha_1 \alpha_2 \ldots \alpha_s\}$, on the understanding that each transposition $(\alpha_h \alpha_k)$ is to be replaced by the product

$$(a_{(h-1)r+1}\, a_{(k-1)r+1})(a_{(h-1)r+2}\, a_{(k-1)r+2}) \ldots (a_{hr} a_{kr}).$$

The expression HG depends in no way on the form of f; so that, if ψ is any function homogeneous and linear in each of the sets of variables $a_1, a_2, \ldots, a_n$, then $HG\psi$ represents a function of the coefficients of the original quantics of the same degree in each as F.

Consider $\{a_1 a_2 \ldots a_n\} f_1$: this is a sum of terms each of which is a term of the n-ic

$$a_{1_x} a_{2_x} \ldots a_{n_x} = a_x.$$

Consider next the expression

$$\{a_3 \ldots a_n\}\{a_1 a_2\}' f_1:$$

this is a sum of terms of $\quad {}_1a_{1x}^{n-2}\, {}_2a_{2x},$

where
$$ {}_1x = x, \quad {}_2x = (xy) = \left\| \begin{matrix} x_1 x_2 \dots x_q \\ y_1 y_2 \dots y_q \end{matrix} \right\|, $$
$$ {}_1a_{1x}^{n-2} = a_{3_x} a_{4_x} \dots a_{n_x}, $$
$$ {}_2a_{{}_2x} = (a_1 a_2)_{{}_2x} = \begin{vmatrix} a_{1,1}\, a_{1,2} \\ a_{2,1}\, a_{2,2} \end{vmatrix} \begin{vmatrix} x_1 x_2 \\ y_1 y_2 \end{vmatrix} + \dots . $$

For, if $a_{1,r}\, a_{2,s}\, B$ is a term of f_1, then $\begin{vmatrix} a_{1,r}\, a_{1,s} \\ a_{2,r}\, a_{2,s} \end{vmatrix} B$ is the corresponding term of $\{a_1 a_2\}' f_1$. More generally, suppose that N consists of n_1 groups degree 1, n_2 groups degree 2, ..., n_k groups degree k; $k \not> q$. Then PNf_1 is a sum of multiples of terms of
$$ (P)\ {}_1a_{1x}^{n_1}\, {}_2a_{2x}^{n_2} \dots {}_ka_{kx}^{n_k} $$
where
$$ {}_1a_{1x}^{n} = a_{1_x} a_{2_x} \dots a_{n_{1_x}}, $$
$a_1, a_2, \dots, a_{n_1}$ being the letters which occur in the groups of degree 1 in N; where
$$ {}_2a_{{}_2x}^{n_2} = \Pi\, (ab)_{{}_2x}, $$
$\Pi\, \{ab\}'$ being the product of groups degree 2 in N, and so on. For, if we replace (ab) by C—a new kind of variable—we obtain n_2 letters $c_1, c_2, \dots, c_{n_2}$, and from the form of P we obtain
$$ P\phi\, (c_1, c_2, \dots, c_n) = \frac{1}{n_2!} P \{c_1 c_2 \dots c_{n_2}\}\, \phi\, (c_1, c_2, \dots, c_{n_2}). $$
And hence, if ϕ is linear in each set of variables c, $P\phi$ may in general be obtained by polarization from a function degree n_2 in a single set of variables c.

The groups of N degrees 3, 4, ... may be dealt with in the same way. If N contain a group degree $> q$, we know by § 8 of my former paper that $Nf = 0$.

The expressions ${}_1a_{1x}, {}_2a_{2x}, \dots, {}_ka_{kx}$, if the sets of variables $a_1, a_2, \dots$ are all cogredient with each other, and contragredient to the sets $x, y, z, \dots$, are unaltered by a linear transformation.

The expression
$$ {}_1a_{1x}^{n_1}\, {}_2a_{2x}^{n_2} \dots {}_ka_{kx}^{n_k} = E $$
is then a concomitant of the forms
$$ a_{1_x}, a_{2_x}, \dots, a_{n_x}. $$

If we operate on E with HG, it will become the symbolical repre-

sentation of a concomitant of the q-ary quantics. Hence $HGPNf_1$, and therefore $HGTf_1$, represents a sum of numerical multiples of the coefficients of certain concomitants of the quantics from which we started. But

$$f = HGf_1 = \Sigma AHGTf_1.$$

Hence any rational integral algebraic function of the coefficients of certain q-ary quantics may be linearly expressed in terms of coefficients of concomitants of these quantics.

21. If f be itself a concomitant, then $HGPNf$ is invariantive. But $HGPNf$ has just been shown to be a linear function of the coefficients of a certain concomitant. Let us write

$$HGPNf = \lambda_0 A_0 + \lambda_1 A_1 + \ldots,$$

where A_0, A_1, ... are the coefficients referred to, and the λ's certain quantities independent of $a_1, a_2, \ldots, a_n$. Now make any linear transformation: the coefficients A_0, A_1, ... are not left unchanged, but the whole expression is left unchanged; hence the quantities $\lambda_0, \lambda_1, \ldots$ must be contragredient to $A_0, A_1, \ldots$; no relation can exist between different coefficients A, since the weight of each is different. The quantities λ are independent of the quantities $a_1, a_2, \ldots, a_n$, and are functions of the remaining variables $x, y, z, \ldots$, defined as being contragredient to the coefficients of A_0, A_1, ... in the concomitant

$${}_1a_{1x}^{n_1}\, {}_2a_{2x}^{n_2} \ldots {}_ka_{kx}^{n_k}.$$

Consider any one of them, and polarize it so as to make it linear in each of the sets of variables it contains. Let us call the function in this form Λ, and use the letters $y_1, y_2, \ldots, y_m$ to denote the various sets of variables which it contains.

Operate on Λ with the series

$$1 = \Sigma AT.$$

Then we may consider separately expressions

$$P_1 N_1 \Lambda.$$

Now, if s be any substitution affecting the letters $y_1, y_2, \ldots$, the expression $s\Lambda$ is cogredient with Λ.

Hence $P_1 N_1 \Lambda$ is cogredient with Λ or zero. But $P_1 N_1 \Lambda$ is obtained by polarization from terms

$${}_1x_{r_1}^{m_1}\, {}_2x_{r_1}^{m_2} \ldots {}_kx_{r_k}^{m_k},$$

where m_1 is the number of groups of N degree 1, and so on. Hence Λ is cogredient with

$$ {}_1x^{m_1}\,{}_2x^{m_2}\,\dots\,{}_kx^{m_k}. $$

Therefore this term is cogredient with

$$ {}_1x^{n_1}\,{}_2x^{n_2}\,\dots\,{}_kx^{n_k}. $$

But it is obvious that two such variable expressions cannot be cogredient unless $m_1 = n_1$, $m_2 = n_2$, ..., for the numbers of terms of ${}_1x$, ${}_2x$, ... are different; so that in any other case it would not be possible to make the terms of the two expressions correspond.

Hence, if P_1N_1 belong to any other expression T except that one which is such that each term contains n_1 negative symmetric groups degree 1, n_2 of degree 2, and so on, then

$$ P_1N_1\Lambda = 0. $$

Hence Λ itself may be obtained by polarization from

$$ {}_1x^{n_1}\,{}_2x^{n_2}\,\dots\,{}_kx^{n_k}. $$

Therefore the concomitant $HGPNf$ is obtained by polarization from

$$ {}_1a^{n_1}_{{}_1x}\,{}_2a^{n_2}_{{}_2x}\,\dots\,{}_ka^{n_k}_{{}_kx}. $$

But
$$ f = \frac{1}{\rho}HGf = \frac{1}{\rho}\,\Sigma AHGTf, $$

where $(HG)^2 = \rho HG$—a relation due to the form of HG.

Hence every concomitant of a system of q-ary quantics can be expressed symbolically as a sum of symbolical products, the factors being of the forms

$$ a_{1,x} = \Sigma a_{1,r}x_r,\ (a_1a_2)_{s,r} = \Sigma \begin{vmatrix} a_{1,r} & a_{1,s} \\ a_{2,r} & a_{2,s} \end{vmatrix} \begin{vmatrix} x_r & x_s \\ y_r & y_s \end{vmatrix},\ \&c., $$

and, finally,
$$ (a_1a_2\dots a_q) = \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,q} \\ a_{2,1} & a_{2,2} & \dots & a_{2,q} \\ \dots & \dots & & \dots \\ a_{q,1} & a_{q,2} & \dots & a_{q,q} \end{vmatrix}. $$

The variables ${}_1x$, ${}_2x$, ... may be regarded as entirely new variables defined as being cogredient with x, (xy), (xyz), ...

22. In every case except the last a linear transformation leaves the symbolical factor unaltered. In the last case the transformed factor

is equal to the original factor multiplied by the modulus of transformation. This constitutes a proof of the fact that, if an algebraic function of the coefficients and variables of certain q-ary quantics is unaltered by any linear transformation, except for a factor which contains only the coefficients of transformation, then this factor must be a power of the modulus of transformation.

V. *Particular Application to Invariants of Binary Forms.*

23. Let I be any invariant of certain binary forms; we may suppose it to be represented symbolically in the ordinary way. By polarization this may be represented as a sum of symbolical products containing each symbolical letter linearly.

Let one such term be*

$$\begin{aligned} i &= (\mathrm{a}_1\mathrm{b}_1)(\mathrm{a}_2\mathrm{b}_2)\ldots(\mathrm{a}_m\mathrm{b}_m) \\ &= \frac{1}{2^m}\{a_1b_1\}'\{a_2b_2\}'\ldots\{a_mb_m\}'(\mathrm{a}_1\mathrm{b}_1)(\mathrm{a}_2\mathrm{b}_2)\ldots(\mathrm{a}_m\mathrm{b}_m). \end{aligned}$$

Operate on this with the series

$$1 = \Sigma AT;$$

then every term Ti before $T_{2,2,\ldots,2}\,i$ vanishes, owing to the existence of the factor of i,
$$\{a_1b_1\}'\ldots\{a_mb_m\}'.$$

Every term Ti after this factor vanishes, because the sets of variables $a_1, b_1, \ldots$ have only two variables in each set; and therefore a negative symmetric group of degree three must annihilate i. ($T_3 i = 0$ owing to the fundamental identity between symbolical factors—which is the same thing as the above statement.) Hence

$$\begin{aligned} i &= A_{2,2,\ldots,2}\,T_{2,2,\ldots,2}\,i \\ &= \frac{1}{2^m}\sqrt{A_{2,2,\ldots,2}}.\{a_1b_1\}'\ldots\{a_mb_m\}'\begin{Bmatrix} a_1a_2\ldots a_m \\ b_1b_2\ldots b_m\end{Bmatrix}(\mathrm{a}_1\mathrm{b}_1)(\mathrm{a}_2\mathrm{b}_2)\ldots(\mathrm{a}_m\mathrm{b}_m). \end{aligned}$$

24. Let us write

$$\begin{aligned} \begin{Bmatrix} a_1a_2\ldots a_m \\ b_1b_2\ldots b_m\end{Bmatrix}(\mathrm{a}_1\mathrm{b}_1)\ldots(\mathrm{a}_m\mathrm{b}_m) &= \begin{Bmatrix} a_1a_2\ldots a_m \\ b_1b_2\ldots b_m\end{Bmatrix}_1 \\ &= (-1)^m\begin{Bmatrix} b_1b_2\ldots b_m \\ a_1a_2\ldots a_m\end{Bmatrix}_1. \end{aligned}$$

* Roman letters are here used to distinguish algebraic symbolical factors from substitutions.

This expression may be looked upon as an algebraic function; so that, if we multiply it by any substitution, the result may be obtained by an interchange of the letters. Otherwise a substitution which multiplies it may be regarded as multiplying the substitutional part PN of the expression. Hence

$$\left\{\begin{matrix} a_2 a_1 a_3 \dots a_m \\ b_1 b_2 b_3 \dots b_m \end{matrix}\right\}_1 = (a_1 a_2) \left\{\begin{matrix} a_1 a_2 a_3 \dots a_m \\ b_1 b_2 b_3 \dots b_m \end{matrix}\right\}_1$$

$$= \left\{\begin{matrix} a_1 a_2 a_3 \dots a_m \\ b_1 b_2 b_3 \dots b_m \end{matrix}\right\}_1 ;$$

so that the letters in either row may be interchanged without altering the value of the function. The fundamental identity has been used completely and no longer need be remembered, for the form of the function is such that it puts in evidence the fact that it is annihilated by any negative symmetric group of degree greater than three. The expression, however, satisfies an identity, for, by § 10,

$$\{a_1 a_2 \dots a_m b_r\} \left\{\begin{matrix} a_1 a_2 \dots a_m \\ b_1 b_2 \dots b_m \end{matrix}\right\}_1 = 0$$

and

$$\{b_1 b_2 \dots b_m a_r\} \left\{\begin{matrix} a_1 a_2 \dots a_m \\ b_1 b_2 \dots b_m \end{matrix}\right\}_1 = 0 ;$$

and, by § 12, this expression satisfies no other relation.

25. Expanding the product of § 23,

$$\{a_1 b_1\}' \{a_2 b_2\}' \dots \{a_m b_m\}',$$

we may obtain $\quad i = (\mathrm{a}_1 \mathrm{b}_1)(\mathrm{a}_2 \mathrm{b}_2) \dots (\mathrm{a}_m \mathrm{b}_m)$

as a sum of expressions such as

$$\left\{\begin{matrix} a_1 a_2 \dots a_m \\ b_1 b_2 \dots b_m \end{matrix}\right\}_1 .$$

Since I is an invariant of certain binary forms,

$$I = HGi,$$

where G is the product of certain positive symmetric groups and H is an expression which interchanges some of these groups. Hence every invariant of a system of binary quantics may be expressed in terms of the forms

$$HG \left\{\begin{matrix} a_1 a_2 \dots a_m \\ b_1 b_2 \dots b_m \end{matrix}\right\}_1 .$$

26. Consider in particular the invariants of a single binary n-ic. We must take the expression

$$HG\left\{\begin{matrix} a_1 a_2 \dots a_m \\ b_1 b_2 \dots b_m \end{matrix}\right\}_1,$$

where the $2m$ letters are arranged in sets of n, and G consists of the product of the positive symmetric groups of these sets.

H is a substitutional expression which interchanges these sets in all possible ways.

Let δ be the degree of the invariant; then

$$2m = \delta n,$$

and m is the weight.

Let us suppose that of these δ sets there are α_0 such that all the letters of the set are in the lower row of the above expression; α_1 such that 1 letter is in the upper row and $n-1$ in the lower row; and so on. In general, let there be α_r sets having r letters in the upper and $n-r$ in the lower row. These numbers completely define the expression, for the letters of either row may be interchanged in all possible ways, the letters of any one set possess no individuality, and no set as a whole possesses individuality.

We may conveniently use the notation

$$f(\alpha_0, \alpha_1, \dots, \alpha_n)$$

to denote this expression.

Every invariant of the n-ic may be expressed linearly in terms of invariants

$$f(\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n).$$

The numbers $\alpha_0, \alpha_1, \dots, \alpha_n$ must satisfy the relations

$$\alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_n = \delta,$$

$$\alpha_1 + 2\alpha_2 + \dots + n\alpha_n = n\alpha_0 + (n-1)\,\alpha_1 + \dots + \alpha_{n-1} = m,$$

the number of letters in either row.

27. These invariants satisfy certain identical relations. We have seen in § 24 that

$$[1 + (a_1 b_r) + (a_2 b_r) + \dots + (a_m b_r)]\left\{\begin{matrix} a_1 a_2 \dots a_m \\ b_1 b_2 \dots b_m \end{matrix}\right\}_1 = 0.$$

Suppose that b_r is a letter of a set which has r letters in the upper row; then b_r is moved up to the upper row, and the letters of the upper row take its place in all possible ways.

If a letter of one of those sets which have s letters above is moved down, then the number a_s is diminished by unity, and the number a_{s-1} increased by unity. At the same time the number a_r is diminished by unity, and the number a_{r+1} increased by unity. But a letter of those sets which have s letters above may be chosen in sa_s ways; hence the identity will be of the form

$$\begin{aligned}\sum_{s\neq r, r+1} sa_s f(a_0, a_1, \ldots, a_{s-1}+1, a_s-1, \ldots, a_r-1, a_{r+1}+1, \ldots, a_n)\\ +r(a_r-1) f(a_0, a_1, \ldots, a_{r-1}+1, a_r-2, a_{r+1}+1, \ldots, a_n)\\ +(r+1)(a_{r+1}+1) f(a_0, a_1, \ldots, a_{r-1}, a_r, a_{r+1}, \ldots, a_n) = 0.\end{aligned}$$

If in this we write a_r+1 for a_r and $a_{r+1}-1$ for a_{r+1} we obtain the identity
$$\sum_s sa_s f(a_0, a_1, a_2, \ldots, a_{s-1}+1, a_s-1, \ldots, a_n) = 0.$$

In the same way, from the identity

$$[1+(a_r b_1)+(a_r b_2)+\ldots+(a_r b_m)] \left\{\begin{matrix} a_1 a_2 \ldots a_m \\ b_1 b_2 \ldots b_m \end{matrix}\right\}_1 = 0,$$

we obtain

$$\sum_s (n-s)\, a_s f(a_0, a_1, a_2, \ldots, a_{s-1}, a_s-1, a_{s+1}+1, \ldots, a_n) = 0.$$

These two are the only linear relations between the invariants of a particular degree, unless we include the following

$$\left\{\begin{matrix} a_1 a_2 \ldots a_m \\ b_1 b_2 \ldots b_m \end{matrix}\right\}_1 = (-1)^m \left\{\begin{matrix} b_1 b_2 \ldots b_m \\ a_1 a_2 \ldots a_m \end{matrix}\right\},$$

and hence $\quad f(a_0, a_1, \ldots, a_n) = (-1)^m f(a_n, a_{n-1}, \ldots, a_0).$

The second of the above identities may be written

$$\begin{aligned} na_0 f(a_0-1, a_1+1, a_2, \ldots, a_n)&\\ &= -(n-1)\, a_1 f(a_0, a_1-1, a_2+1, \ldots, a_n)\\ &\quad -(n-2)\, a_2 f(a_0, a_1, a_2-1, a_3+1, \ldots, a_n)\\ &\quad -\ldots; \end{aligned}$$

so that, if $a_1 \neq 0$, the invariant $f(a_0, a_1, \ldots, a_n)$ may be expressed in terms of invariants in which the value of a_0 is increased, of a_1 is decreased, while that of a_n is not decreased.

Similarly, if $a_{n-1} \neq 0$, we may express this invariant in terms of invariants in which a_n is increased, a_{n-1} decreased, and a_0 is not decreased.

Proceeding step by step, we may decrease a_1 and a_{n-1} to zero, at the

same time increasing α_0 and α_n. Thus every invariant may be expressed in terms of such as have α_1 and α_{n-1} zero.

Hence, if $n<4$, there is not more than one invariant for each degree.

28. Although the above identities contain all the linear relations between the invariants f, yet there are other relations—not independent of these—which it is useful to have.

Consider the result of multiplying

$$\begin{Bmatrix} a_1 a_2 \ldots a_m \\ b_1 b_2 \ldots b_m \end{Bmatrix}_1 = I$$

by any positive symmetric group Γ. We may regard the product as the sum of the results of operating on I with each of the substitutions of Γ; or else we may multiply the substitutional part of I by Γ as explained (Section III.), and so express the product as a sum of such terms as I, but each of which have all the letters of Γ in the same row; the two expressions for the product must be equal.

In particular, if the degree of Γ is greater than m, then the second expression is zero.

Consider now the invariant

$$f(\alpha_0, \alpha_1, \ldots, \alpha_n) = HGI,$$

and let us find the result of writing ΓI for I. Let the degree of Γ be ϖ, and let us suppose that the letters of Γ occupy ρ places in the upper row of I, and $\varpi-\rho$ in the lower row. Then ΓI is the sum of the results of arranging the ϖ letters of Γ in all possible ways in the ϖ places assigned to them in I. Of these $\binom{\varpi}{\rho}$ are different arrangements.

The letters of I will be supposed to be distributed in the manner defined by certain numbers,

$$\alpha_{0,0}, \alpha_{0,1}, \ldots, \alpha_{0,n};\ \alpha_{1,0}, \ldots, \alpha_{1,n-1};\ \ldots;\ \alpha_{n,0},$$

where the number $\alpha_{\lambda,\mu}$ signifies that there are $\alpha_{\lambda,\mu}$ sets of letters such that μ letters from each set appear in Γ, λ appear in the upper row of I but not in Γ, and the rest, $n-\lambda-\mu$ in number, appear in the lower row of I but not in Γ.

We must consider some one term of the sum $HG\Gamma I$: this will be defined completely when the positions of the letters of Γ are given. These will be given by the numbers $\alpha_{\lambda,\mu,\nu}$, the meaning of which symbol is that of the $\alpha_{\lambda,\mu}$ sets of letters defined as above there are

$a_{\lambda,\mu,\nu}$ in the particular expression under consideration which have ν of the μ letters belonging to Γ in the upper row and the remaining $\mu-\nu$ in the lower row.

Now the $a_{\lambda,\mu}$ sets of letters can be divided up into $a_{\lambda,\mu,0}, a_{\lambda,\mu,1}, \ldots, a_{\lambda,\mu,\mu}$ sets (of which the sum must be $a_{\lambda,\mu}$), in $\dfrac{a_{\lambda,\mu}!}{a_{\lambda,\mu,0}!\, a_{\lambda,\mu,1}! \ldots a_{\lambda,\mu,\mu}!}$ ways.

Further, in one of the $a_{\lambda,\mu,\nu}$ sets the ν letters which are to appear in the upper row of the invariant may be chosen in $\binom{\mu}{\nu}$ ways. Hence the number of different arrangements of the members of these $a_{\lambda,\mu}$ sets which will fulfil the requirements of the term considered is

$$\frac{a_{\lambda,\mu}!}{a_{\lambda,\mu,0}!\, a_{\lambda,\mu,1}! \ldots a_{\lambda,\mu,\mu}!} \binom{\mu}{0}^{a_{\lambda,\mu,0}} \binom{\mu}{1}^{a_{\lambda,\mu,1}} \ldots \binom{\mu}{\mu}^{a_{\lambda,\mu,\mu}}.$$

Hence the total number of such terms is

$$\prod_{\lambda=0}^{\lambda=n-1} \prod_{\mu=1}^{\mu=n-\lambda} \left\{ a_{\lambda,\mu}! \prod_{\nu=0}^{\nu=\mu} \frac{\binom{\mu}{\nu}^{a_{\lambda,\mu,\nu}}}{a_{\lambda,\mu,\nu}!} \right\}.$$

Consider then the coefficient of the term $f(\gamma_0, \gamma_1, \ldots, \gamma_n)$: it is obtained as the sum of the coefficients of the terms for which the quantities $a_{\lambda,\mu,\nu}$ satisfy the relations

$$\sum_{\lambda=0}^{\lambda=p} \sum_{\mu=p-\lambda}^{\mu=n-\lambda} a_{\lambda,\mu,p-\lambda} = \gamma_p.$$

This sum may be seen at once to be the coefficient of $x_0^{\gamma_0} x_1^{\gamma_1} \ldots x_n^{\gamma_n}$ in the expansion of

$$x_0^{a_{0,0}} x_1^{a_{1,0}} \ldots x_n^{a_{n,0}},$$

$$(x_0+x_1)^{a_{0,1}} (x_1+x_2)^{a_{1,1}} \ldots (x_{n-1}+x_n)_{a^{n-1,}},$$

$$\left(x_0+\binom{2}{1} x_1+x_2\right)^{a_{0,2}} \left(x_1+\binom{2}{1} x_2+x_3\right)^{a_{1,2}} \ldots \left(x_{n-2}+\binom{2}{1} x_{n-1}+x_n\right)^{a_{n-2,2}},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\left(x_0+\binom{n}{1} x_1+\binom{n}{2} x_2+\ldots+x_n\right)^{a_{0,n}}.$$

This coefficient has only taken into account the different terms of ΓI, but each term is repeated $\rho!\,(\varpi-\rho)!$ times; the coefficient just found must then be multiplied by this.

It is interesting to notice that, if we write

$$x_0 = 1,\ x_1 = x,\ x_2 = x^2,\ \ldots,\ x_n,$$

then the coefficient of x^w, where w is the weight of the invariant considered, in the above expansion is $\binom{\varpi}{\rho}$. This is as it should be, since this coefficient represents the sum of the coefficients which have to be considered.

29. To find when $\varpi \ngtr m$ the second form in which the expression $HG\Gamma I$ may be written.

Let us write
$$I = \begin{Bmatrix} a_1 a_2 \dots a_m \\ b_1 b_2 \dots b_m \end{Bmatrix}_1,$$
$$\Gamma = \{a_1 a_2 \dots a_\rho b_1 b_2 \dots b_{\varpi-\rho}\}.$$

Then
$$\begin{aligned}\Gamma I = \rho!\,&[1+(b_{\varpi-\rho}a_1)+(b_{\varpi-\rho}a_2)+\dots+(b_{\varpi-\rho}b_{\varpi-\rho-1})]\\ &\times[1+(b_{\varpi-\rho-1}a_1)+\dots+(b_{\varpi-\rho-1}b_{\varpi-\rho-2})]\\ &\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\\ &\times[1+(b_1a_1)+(b_1a_2)+\dots+(b_1a_\rho)]\\ &\times\begin{Bmatrix} a_1 a_2 \dots a_m \\ b_1 b_2 \dots b_m \end{Bmatrix}_1.\end{aligned}$$

Now
$$\begin{aligned}[1+(b_1a_1)+\dots+(b_1a_\rho)]&\begin{Bmatrix} a_1 a_2 \dots a_m \\ b_1 b_2 \dots b_m \end{Bmatrix}_1\\ &= -[(b_1a_{\rho+1})+(b_1a_{\rho+2})+\dots+(b_1a_m)]\begin{Bmatrix} a_1 a_2 \dots a_m \\ b_1 b_2 \dots b_m \end{Bmatrix}_1.\end{aligned}$$

Taking each term we obtain
$$\Gamma I = (-)^{\varpi-\rho}\rho!\,\Sigma I', \tag{IV.}$$
where I' has all the letters of Γ in the upper row, and where the places which $b_1 b_2 \dots b_{\varpi-\rho}$ occupy in the lower row of I are filled up with $\varpi-\rho$ of the letters $a_{\rho+1} \dots a_m$, in all possible ways. The number of *different* ways in which this can be done is $\binom{m-\rho}{\varpi-\rho}$, but the number of terms in the $\Sigma I'$ of equation (IV.) is
$$(m-\rho)(m-\rho-1)\dots(m-\varpi+1) = \binom{m-\rho}{\varpi-\rho}(\varpi-\rho)!.$$

Hence each different term is affected by the coefficient
$$(-)^{\varpi-\rho}(\varpi-\rho)!\,\rho!.$$

Hence
$$I = (-1)^{\varpi-\rho}(\varpi-\rho)!\,\rho!\,\Sigma\lambda f(\beta_0, \beta_1, \dots, \beta_n),$$

where the coefficient λ is obtained in the same way as the corresponding coefficient of the last paragraph; it is equal to the coefficient of $x_0^{\beta_0}x_1^{\beta_1}\dots x_n^{\beta_n}$ in the expansion of

$$x_0^{a_{0,0}}x_1^{a_{0,1}}x_2^{a_{0,2}}\dots x_n^{a_{0,n}},$$

$$(x_0+x_1)^{a_{1,0}}(x_1+x_2)^{a_{1,1}}\dots(x_{n-1}+x_n)^{a_{1,n-1}},$$

$$\left(x_0+\binom{2}{1}x_1+x_2\right)^{a_{2,0}}\left(x_1+\binom{2}{1}x_2+x_3\right)^{a_{2,1}}\dots\left(x_{n-2}+\binom{2}{1}x_{n-1}+x_n\right)^{a_{2,n-2}},$$

$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$

$$\left(x_0+\binom{n}{1}x_1+\binom{n}{2}x_2+\dots+x_n\right)^{a_{n,0}}.$$

Let us call this coefficient $\phi_2(\beta_0, \beta_1, \dots, \beta_n)$, and the corresponding coefficient of the last paragraph $\phi_1(\gamma_0, \gamma_1, \dots, \gamma_n)$; then we obtain the identity

$$\Sigma\phi_1(\gamma_0, \gamma_1, \dots, \gamma_n)f(\gamma_0, \gamma_1, \dots, \gamma_n)$$
$$= (-1)^{\varpi-\rho}\,\Sigma\phi_2(\beta_0, \beta_1, \dots, \beta_n)f(\beta_0, \beta_1, \dots, \beta_n).$$

30. Let us consider the case

$$a_{\lambda,\mu} = 0, \quad \text{unless} \quad \mu = 0 \text{ or } \lambda = 0,$$

and $$a_{0,\mu} = 0, \quad \text{unless} \quad \mu = 0 \text{ or } n.$$

Then $$\varpi = na_{0,n},$$

$$m-\rho = \Sigma\lambda a_{\lambda,0}.$$

The two sides of the identity will be given by the expansions

$$x_0^{a_{0,0}}x_1^{a_{1,0}}\dots x_n^{a_{n,0}}\left(x_0+nx_1+\binom{n}{2}x_2+\dots+x_n\right)^{a_{0,n}}$$

and $$(-1)^{\varpi-\rho}x_0^{a_{0,0}}x_n^{a_{0,n}}(x_0+x_1)^{a_{1,0}}(x_0+2x_1+x_2)^{a_{2,0}}\dots$$
$$\dots(x_0+nx_1+\dots+x_n)^{a_{n,0}}.$$

Any term of either expansion

$$\lambda x_0^{\gamma_0}x_1^{\gamma_1}\dots x_n^{\gamma_n},$$

where $$\gamma_1+2\gamma_2+\dots+n\gamma_n = m,$$

represents a term of the identity, viz.,

$$\lambda f(\gamma_0, \gamma_1, \dots, \gamma_n).$$

Now it will be regarded as a reduction when $f(a_0, a_1, \dots, a_n)$ is

expressed in terms of invariants in which one or both of a_0, a_n is increased; also if this invariant is expressed in terms of invariants $f(\gamma_0, \gamma_1, \ldots, \gamma_n)$ such that the first argument γ_2, which differs from the corresponding argument a_r, exceeds it, whether we start comparing them at one end or at the other. In such a case we shall say that $f(\gamma_0, \gamma_1, \ldots, \gamma_n)$ is more simple than $f(a_0, a_1, \ldots, a_n)$.

Let us suppose that $$a_{0,n} > a_{n,0}.$$

Now $$a_{1,0}+2a_{2,0}+\ldots+na_{n,0} = m-\rho.$$

Hence $x_0^{\beta_0+a_{0,0}}x_1^{\beta_1+a_{1,0}}\ldots x_n^{\beta_n+a_{n,0}}$, a term of the expansion of

$$x_0^{a_{0,0}}x_1^{a_{1,0}}\ldots x_n^{a_{n,0}}(x_0+nx_1+\ldots+x_n)^{a_{0,n}},$$

will give an invariant

$$\beta_1+2\beta_2+\ldots+n\beta_n = \rho$$

and $$\beta_0+\ \beta_1+\ldots+\ \beta_n = a_{0,n}.$$

Let us suppose that $$\rho = ra_{0,n};$$

then $$\binom{n}{r}^{a_{0,n}} f(a_{0,0},\ a_{1,0},\ \ldots,\ a_{r,0}+a_{0,n},\ a_{r+1,0},\ \ldots,\ a_{n,0})$$

is a term of the left-hand side of the identity. Every other term on this side is more simple; if not, we must be able to choose positive numbers $\beta_0, \beta_1, \ldots, \beta_r$, such that

$$\beta_0+\ \beta_1+\ldots+\ \beta_r = a_{0,n}$$

and $$\beta_1+2\beta_2+\ldots+r\beta_r = ra_{0,n};$$

or else positive numbers $\beta_r, \beta_{r+1}, \ldots, \beta_n$

such that $$\beta_r+\beta_{r+1}+\ldots+\beta_n = a_{0,n},$$

$$r\beta_r+(r+1)\,\beta_{r+1}+\ldots+n\beta_n = ra_{0,n}.$$

The only solution in either case is $\beta_r = a_{0,n}$. By hypothesis the number $a_{0,n} > a_{n,0}$, and so all the terms on the other side of the identity are more simple than that under consideration.

Hence, if $a_r > a_n$, we may express $f(a_0, a_1, \ldots, a_n)$ in terms of simpler invariants.

In the same way we may deduce from this identity that, if $a_r+a_{r+1} > a_n$, the invariant $f(a_0, a_1, \ldots, a_n)$ may be expressed in terms of simpler invariants.

These reductions will evidently be true sometimes when

$$a_r+a_{r+1} = a_n.$$

Now $\quad f(a_0, a_1, \ldots, a_n) = (-1)^m f(a_n, a_{n-1}, \ldots, a_0)$.

Hence we may perform a similar reduction, if $a_r + a_{r+1} > a_0$.

31. Let us consider some particular cases.

For the quartic we may have $a_1 = 0$, $a_3 = 0$, and $a_2 \ngtr a_0$ or a_4. The invariants are evidently

$$f(1, 0, 0, 0, 1), \quad f(1, 0, 1, 0, 1), \quad f(2, 0, 0, 0, 2),$$

and so on.

For the quintic we may take $a_1 = 0$, $a_4 = 0$, $a_2 + a_3 \ngtr a_0$ or a_4. Let us find the degree of the lowest *gauche* invariant. For a *gauche* invariant the weight m is odd; hence

$$f(a_0, a_1, \ldots, a_n) = -f(a_n, a_{n-1}, \ldots, a_0).$$

Hence, if $\quad a_0 = a_n, \; a_1 = a_{n-1}, \; \ldots,$

$$f(a_0, a_1, \ldots, a_n) = 0.$$

Now $\quad m = a_1 + 2a_2 + \ldots + na_n = na_0 + (n-1)\, a_1 + \ldots + a_{n-1}$;

and therefore $\quad n(a_0 - a_n) + (n-2)(a_1 - a_{n-1}) + \ldots = 0$.

For the quintic, if $\quad a_1 = 0 = a_4$,

$$5(a_0 - a_5) + (a_2 - a_3) = 0.$$

Hence for the lowest *gauche* invariant

$$a_3 - a_2 = 5(a_0 - a_5) \neq 0.$$

Also we may take $a_2 + a_3 \ngtr a_0$ or a_5; hence, if $a_3 > a_2$, $a_0 > 5$.

Consider $\quad a_2 = 0, \quad a_3 = 5, \quad a_0 = 6, \quad a_5 = 5.$

This is the first case in which the equations

$$a_0 = a_5, \quad a_2 = a_3$$

need not be satisfied; but it does not give a *gauche* invariant. The earliest case is of degree 18, there being only one invariant of this degree, viz.,

$$f(7, 0, 0, 5, 0, 6).$$

VI. *The Concomitants of a single q-ary Quadratic.*

32. We have already proved that any function of the coefficients of a q-ary quantic may be expressed as a linear function of the coefficients of concomitants of the form $PN\phi$.

When this function is itself a concomitant then the function of the

coefficients becomes $PN\phi$ itself. The manner in which the variables appear is known from the form of PN.

Hence the expression

$$\left\{\begin{array}{l} a_{1,1}\, a_{1,2} \,\cdots\cdots\cdots\, a_{1,\beta_1} \\ a_{2,1}\, a_{2,2} \,\cdots\, a_{2,\beta_2} \\ \cdots \quad\; \cdots \quad\; \cdots \\ a_{k,1} \,\cdots\, a_{k,\beta_k} \end{array}\right\}$$

defines a concomitant; and all concomitants may be expressed in terms of these.

33. For the case of a quadratic, these concomitants are operated upon by an operator G which connects the letters together two and two. For a single quadratic we have further an operator H which interchanges the sets of two in all possible ways. The concomitant will be completely defined when we know the numbers $a_{1,1}$ of sets having both letters in the first row. $a_{2,2}$ of sets having both letters in the second row, $a_{1,2}$ of sets having one letter in the first and one in the second row, and so on.

We will write the concomitant then in the form

$$f\,(a_{1,1},\ a_{2,2},\ \ldots,\ a_{k,k}\,;\ \ a_{1,2},\ \ldots,\ a_{k-1,k}),$$

where $a_{r,s}$ is the number of sets of letters having one letter in the r-th and one in the s-th row.

Consider the identity corresponding to

$$\{a_{1,1}\, a_{1,2}\, \ldots\, a_{1,\beta_1} b\}\ I = 0,$$

where b is a letter of $a_{1,2}$. We obtain

$$\begin{aligned} 2\,(a_{1,1}+1)\,f\,(a_{1,1},\ a_{2,2},\ \ldots\,;\ a_{1,2},\ a_{1,3},\ \ldots) \\ +(a_{1,2}-1)\,f\,(a_{1,1}+1,\ a_{2,2}+1,\ \ldots\,;\ a_{1,2}-2,\ a_{1,3},\ \ldots) \\ +\sum_{r=3}^{k} a_{1,r}\,f\,(a_{1,1}+1,\ a_{2,2},\ \ldots\,;\ a_{1,2}-1,\ a_{1,3},\ \ldots,\ a_{1,r}-1,\ \ldots,\ a_{2,r}+1, \\ \ldots,\ a_{k-1,k}) = 0. \end{aligned}$$

Hence, unless $\qquad a_{1,2} = 0 = a_{1,3} \ldots = a_{1,k},$

we can increase $a_{1,1}$. If β_1 is odd, the letters of the top row cannot all be made to belong to the sets $a_{1,1}$; and, by the above identity, the concomitant vanishes.

Similarly, we find by taking the letters of the second row with one

out of a lower row, that $a_{2,r}$ may be taken to be zero: and that the concomitant vanishes unless β_2 is even.

Hence, in order to give a non-zero concomitant, the number of letters in each row of PN must be even. Also we may suppose that there are no sets $a_{r,s}$, $r \neq s$.

There is then only one independent form, viz.,

$$f\left(\frac{\beta_1}{2}, \frac{\beta_2}{2}, \ldots, \frac{\beta_k}{2};\ 0, 0, \ldots, 0\right).$$

34. If there is a product of concomitants of the quadratic which is of the same degree, and of the same order in each kind of variable, as the above, then the above is reducible.

Hence the only irreducible forms for a single q-ary quadratic are given by

$$f(1, 1, \ldots, 1;\ 0, 0, \ldots, 0),$$

where the number of rows of the PN from which the expression is obtained is $\ngtr q$.

Thus the irreducible concomitants of a q-ary quadratic are q in number; they may be written symbolically

$$a_{1x}^2,\ (ab)_{\xi}^2,\ \ldots,\ (a_1 a_2 \ldots a_{q-1})_{q-1x}^2,\ (a_1 a_2 \ldots a_q)^2.$$

On the Series $1+\left(\frac{p}{1}\right)^3+\left\{\frac{p(p+1)}{1.2}\right\}^3+\ldots$ *By* F. Morley.

Received and read April 11th, 1901. Revised* March, 1902.

A curious formula for the sum of the cubes of the coefficients in the Maclaurin series for

$$(1-x)^{-p}$$

for the special case of p a negative integer was proposed by me long since, and was proved by Dixon (*Messenger of Mathematics*,

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* [The title of the paper was changed in the revision: *cf. Proc.*, Vol. xxxiv., p. 49.—Sec.]

Vol. xx.) and by Richmond (*ib.*, Vol. xxi.), and again, as an illustration of generating functions, by MacMahon (*Quarterly Journal*, Vol. xxxiii.).

I wish now to sum the series for p a real positive number; where, by ordinary tests of convergence, we take

$$p < \tfrac{2}{3}.$$

1. The special hypergeometric series of the second kind

$$f(p, t) = 1 + \left(\frac{p}{1}\right)^3 t + \left\{\frac{p(p+1)}{1.2}\right\}^3 t^2 + \ldots \tag{1}$$

is the sum of terms independent of x and y in

$$(1-x)^{-p}(1-y)^{-p}(1-t/xy)^{-p}, \tag{2}$$

where x, y, t are real and positive,

$$x < 1, \quad y < 1, \quad t < xy,$$

and by $(1-x)^{-p}$ we mean

$$1 + \frac{p}{1}x + \frac{p(p+1)}{1.2}x^2 + \ldots,$$

that is

$$\frac{1}{\Gamma p}\sum_0^\infty \frac{\Gamma(p+n)}{\Gamma(1+n)}x^n.$$

Now (2) is

$$(1-x-y+xy)^{-p}(1-t/xy)^{-p}$$
$$= (1+xy)^{-p}(1-t/xy)^{-p}\Sigma\frac{\Gamma(p+m)}{\Gamma p\,\Gamma(1+m)}\left(\frac{x+y}{1+xy}\right)^m.$$

For m odd there is no term independent of x and y, but for m even, say $= 2n$, we have such terms arising from

$$\frac{1}{\Gamma p}\Sigma(1+xy)^{-p-2n}(1-t/xy)^{-p}\frac{\Gamma(p+2n)}{\Gamma(1+2n)}\,\frac{\Gamma(1+2n)}{\Gamma^2(1+n)}(xy)^n.$$

That is, if $z = xy$, so that z is positive and < 1, $f(p, t) =$ sum of terms independent of z in

$$\frac{1}{\Gamma p}\Sigma\frac{\Gamma(p+2n)}{\Gamma^2(1+n)}z^n(1+z)^{-p-2n}\left(1-\frac{t}{z}\right)^{-p}$$
$$= \frac{1}{\Gamma p}\Sigma\frac{\Gamma(p+2n)}{\Gamma^2(1+n)}\,\frac{t^n}{\Gamma p\,\Gamma(p+2n)}$$
$$\times\left[\frac{\Gamma(p+n)\,\Gamma(p+2n)}{\Gamma(1+n)\,\Gamma 1} - \frac{\Gamma(p+n+1)\,\Gamma(p+2n+1)}{\Gamma(2+n)\,\Gamma 2}t + \ldots\right],$$

or $f(p, t) = \dfrac{1}{\Gamma^2 p} \Sigma t^n \dfrac{\Gamma(p+n)\,\Gamma(p+2n)}{\Gamma^3(1+n)} F(p+n, p+2n, 1+n; -t),$ (3)

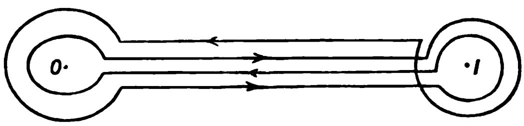
with the usual notation $F(\alpha, \beta, \gamma; x)$ for a hypergeometric series of the first kind.

The series F is such that

$$\gamma = 1-\alpha+\beta;$$

on this depends further progress.

2. We wish to make $t = -1$, but the series F is then useless. We replace then the series by an integral, taken over the path in figure.



The hypergeometric function

$$\int_{(0,1)} x^{\beta-1} (1-x)^{\gamma-\beta-1} (1+tx)^{-\alpha}\, dx$$

taken over this path becomes, when we write

$$\gamma = 1-\alpha+\beta,$$

$$\int_{(0,1)} x^{\beta-1} (1-x)^{-\alpha} (1+tx)^{-\alpha}\, dx,$$

and when $t = 1$ is $\displaystyle\int_{(0,1)} x^{\beta-1} (1-x^2)^{-\alpha}\, dx,$

or, if $x^2 = y$ and the consequent y-path be denoted by $(0^2, 1)$, is

$$\tfrac{1}{2} \int_{(0^2,1)} y^{\frac{1}{2}\beta-1} (1-y)^{-\alpha}\, dy.$$

When x passes from near 0 to near 1 so does y; when x circles 1 y circles 1; but when x circles 0 y circles 0 twice.

Hence, expressing Euler's first integral $B(\beta, \beta')$ on the one hand by

$$-(1-\exp 2\pi i\beta)(1-\exp 2\pi i\beta')\, B = \int_{(0,1)} x^{\beta-1} (1-x)^{\beta'-1}\, dx,$$

on the other hand by $\dfrac{\Gamma\beta\,\Gamma\beta'}{\Gamma(\beta+\beta')}$

(see Klein, *Hypergeometric Function*, p. 142; Schlesinger, *Handbuch*

der Differentialgleichungen, Vol. III., p. 452), we have from the first form

$$\int_{(0^{2},\,1)} y^{\frac{1}{2}\beta-1}(1-y)^{-a}\,dy$$

$$= -B\left(\frac{\beta}{2},\ 1-a\right)(1-\exp 2\pi i\beta)[1-\exp 2\pi i(1-a)]$$

and $\displaystyle\int_{(0,\,1)} x^{\beta-1}(1-x)^{-a}\,dx$

$$= -B(\beta,\ 1-a)(1-\exp 2\pi i\beta)[1-\exp 2\pi i(1-a)].$$

Hence $\displaystyle\lim_{t=1}\int_{(0,\,1)} x^{\beta-1}(1-x)^{-a}(1+tx)^{-a}\,dx$

$$= \tfrac{1}{2}\,\frac{B\left(\frac{\beta}{2},\ 1-a\right)}{B(\beta,\ 1-a)}\int_{(0,1)} x^{\beta-1}(1-x)^{-a}\,dx;$$

and therefore

$$\lim_{t=1} F(a,\beta,1-a+\beta;\ -1) = \tfrac{1}{2}\,\frac{B\left(\frac{\beta}{2},\ 1-a\right)}{B(\beta,\ 1-a)} = \tfrac{1}{2}\,\frac{\Gamma\frac{\beta}{2}\,\Gamma(1-a+\beta)}{\Gamma\beta\,\Gamma\left(1-a+\frac{\beta}{2}\right)}. \tag{4}$$

Hence (3) becomes, writing $a=p+n$, $\beta=p+2n$,

$$f(p,1) = \frac{1}{2\Gamma^2 p}\,\Sigma\,\frac{\Gamma(p+n)\,\Gamma(p+2n)\,\Gamma\left(\frac{p}{2}+n\right)\Gamma(1+n)}{\Gamma^3(1+n)\,\Gamma(p+2n)\,\Gamma\left(1-\frac{p}{2}\right)}$$

$$= \frac{1}{2\Gamma^2 p\,\Gamma\left(1-\frac{p}{2}\right)}\,\Sigma\,\frac{\Gamma(p+n)\,\Gamma\left(\frac{p}{2}+n\right)}{\Gamma^2(1+n)}.$$

But (Forsyth, *Differential Equations*, p. 197)

$$\Sigma\,\frac{\Gamma(p+n)\,\Gamma(q+n)}{\Gamma(1+n)\,\Gamma(r+n)} = \frac{\Gamma p\,\Gamma q\,\Gamma(r-p-q)}{\Gamma(r-p)\,\Gamma(r-q)},$$

when $$r > p+q.$$

Hence $$f(p,1) = \frac{1}{2\Gamma^2 p\,\Gamma\left(1-\frac{p}{2}\right)}\,\frac{\Gamma p\,\Gamma\left(\frac{p}{2}\right)\Gamma\left(1-\frac{3p}{2}\right)}{\Gamma(1-p)\,\Gamma\left(1-\frac{p}{2}\right)}$$

$$= \frac{\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(1-\frac{3p}{2}\right)}{\Gamma(1+p)\,\Gamma(1-p)\,\Gamma^2\left(1-\frac{p}{2}\right)}$$

$$= \frac{p\frac{\pi}{2}}{\sin\frac{p\pi}{2}}\;\frac{\sin p\pi}{p\pi}\;\frac{\Gamma\left(1-\frac{3p}{2}\right)}{\Gamma^3\left(1-\frac{p}{2}\right)};$$

so *that, finally,*

$$1+\left(\frac{p}{1}\right)^3+\left\{\frac{p(p+1)}{1.2}\right\}^3+\ldots = \cos\frac{p\pi}{2}\,\frac{\Gamma\left(1-\frac{3p}{2}\right)}{\Gamma^3\left(1-\frac{p}{2}\right)}. \tag{5}$$

This for p a negative integer $-m$ gives at once the old result that

$$1-\binom{m}{1}^3+\binom{m}{2}^3-\ldots,$$

which is evidently 0 for m odd, is

$$(-)^n\frac{(3n)!}{(n!)^3} \text{ for } m=2n.$$

For any number p real or complex, such that $|p| < \frac{2}{3}$, formula (5) is true.

3. The simplest of some series whose sums follow from (5) are worth noting. We have

$$\log\Gamma(1+x) = -\gamma x+\tfrac{1}{2}s_2x^2-\tfrac{1}{3}s_3x^3+\ldots,$$

where $|x| < 1$, γ is Euler's constant, and

$$s_m = \sum_{n=1}^{\infty}\frac{1}{n^m}.$$

Also
$$\log\cos\frac{\pi}{2} = \sum_1^{\infty}\log\left\{1-\frac{p^2}{(2m-1)^2}\right\},$$

Hence, from (5), taking logarithms and expanding in powers of p,

$$\frac{s_2}{2}(3^2-3)\left(\frac{p}{2}\right)^2+\frac{s_3}{3}(3^3-3)\left(\frac{p}{2}\right)^3+\frac{s_4}{4}(3^4-3)\left(\frac{p}{2}\right)^4+\ldots$$
$$-s_2\left(\frac{1}{2^2}\right)p^2-\frac{s_4}{2}\left(1-\frac{1}{2^4}\right)p^4-\ldots$$

$$= s_3p^3+3p^4\left[\frac{1}{2^3}+(1+\tfrac{1}{2})\frac{1}{3^3}+(1+\tfrac{1}{2}+\tfrac{1}{3})\frac{1}{4^3}+\ldots\right].$$

Hence $$\frac{1}{2^3}+\left(1+\tfrac{1}{2}\right)\frac{1}{3^3}+\left(1+\tfrac{1}{2}+\tfrac{1}{3}\right)\frac{1}{4^3}+\dots$$
$$=\tfrac{1}{3}\,\frac{24}{2^5}\,s_4=\frac{s_4}{4}=\frac{\pi^4}{360}. \qquad (6)$$

Again, our series is the sum of coefficients of all powers of xyz in
$$\{(1-x)(1-y)(1-z)\}^{-p},$$
or in
$$\exp p\left(\sigma_1+\tfrac{1}{2}\sigma_2+\tfrac{1}{3}\sigma_3+\dots\right),$$
where
$$\sigma_n=x^n+y^n+z^n.$$
The part contributed by the coefficient of p^4 is the sum of coefficients of $(xyz)^n$ in
$$\frac{1}{4!}\left(\sigma_1+\frac{\sigma_2}{2}+\frac{\sigma_3}{3}+\dots\right)^4,$$
$$=\frac{6}{2!}\left\{\frac{1}{2.2^3}+\frac{1}{1.2.3^3}+\frac{1}{1.3.4^3}+\frac{1}{1.4.5^3}+\frac{1}{2.3.5^3}+\dots\right\}$$
$$\times\left\{1+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right\},$$
$$=3s_4\left[\sum_3^\infty\frac{1}{P_r r^3}+\tfrac{1}{8}\right],$$

where P_r is the product of any two relative primes whose sum is r.

But the coefficient of p^4 in the logarithm, and hence also in the series (5), was $\frac{3s_4}{4}$. Hence
$$\sum_3^\infty\frac{1}{P_r r^3}=\tfrac{1}{8}.$$

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